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# ELEMENTS

OF

# TRIGONOMETRY,

AND

# TRIGONOMETRICAL ANALYSIS,

PRELIMINARY TO THE DIFFERENTIAL CALCULUS:

FIT FOR THOSE WHO HAVE STUDIED THE PRINCIPLES OF ARITHMETIC
AND ALGEBRA, AND SIX BOOKS OF EUCLID.

BY

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> Tant que l'algèbre et la géométrie ont été séparées, leur progrès ont été lents et leurs usages bornés ; mais lorsque ces deux sciences se sont réunies, elles se sont prêtées des forces mutuelles, et ont marché ensemble d'un pas rapide vers la perfection.—LAGRANGE.

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# PREFACE.

THE three great branches of elementary mathematics, meaning all that should precede the study of the Differential Calculus, are Arithmetic, Algebra, and Geometry. Each pair of these gives rise to a new inquiry, namely, the connexion which exists between the two. Thus, it is practicable to consider separately, 1. The application of arithmetic to algebra, and of algebra to arithmetic. 2. That of arithmetic to geometry, and of geometry to arithmetic. 3. That of algebra to geometry, and of geometry to algebra.

This will, at first sight, appear something like an undue quantity of distinction: nevertheless, with the exception only of the two-fold comparison of arithmetic with geometry, there is in it no degree of separation which cannot be fully justified, either as matter of necessity or convenience.

The application of arithmetic to algebra is made in the very formation of the latter science, though based upon considerations neither known nor admitted in the former. But, nevertheless, those considerations are derived from arithmetic in this sense,—that the want of them is suggested by the imperfections of the latter, and the method of arriving at them by the manner in which those imperfections appear.

The ideas to which arithmetic cannot fail to bring us, are more than its language has power to express; and the passage from that science to algebra consists in the methodical arrangement of the ideas to be expressed, and the invention of a language proper for the purpose. The application of algebra to arithmetic is a department of a more special character. It is of little consequence whether it be made a separate study, or not; the indispensable branches of it appear in their proper places, and nothing more is necessary than to point out the connexion. The higher parts of what is called the theory of numbers have offered, as yet, singularly little aid in the application of mathematics to the sciences of matter, and may, therefore, be omitted entirely by the student whose wishes on this subject are bounded by the possession of an instrument for physical inquiry. Neither will the more exclusively mathematical student find any thing in that theory for which he should make special preparation in his elementary reading: I have, therefore, omitted it altogether in my *Algebra*.

The application of arithmetic to geometry must be made as soon as the study of proportion begins. But, viewed by the side of arithmetic, geometry becomes the science of continuous magnitude in general: that is to say, the considerations on which it is necessary to dwell are such as apply equally to all magnitudes, as well as to spaces or lengths. In the accompanying* treatise, *On the Connexion of Number and Magnitude*, I have endeavoured, at least, to place the real difficulties of the subject before the higher class of students; guaranteeing nothing more than this, that a larger proportion of readers will understand the tract in question than would, by themselves, be able to master the Fifth Book of Euclid. The extension of the arithmetical notion of ratio, (shewn to be necessary, as well as furnished, by the consideration of magnitude in general, but principally of space magnitudes), constitutes the primary portion of the application of geometry to arithmetic.

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* I have placed this treatise at the end of the *Trigonometry*; but it should be understood as intended to be read first, or, at least, that the two should be read together.

The application of algebra to geometry is divisible into two distinct subjects. 1. The science usually called by that name, but which may be styled the theory of curves and surfaces. I may say of this part of the study, that though, on various accounts, it is *very desirable* that it should be made a separate branch, still it is not *indispensable*. A student of more than average intelligence might pick up, as he went along, enough of this part of mathematics to enable him to pursue his career: no new principles are insisted on, and the independent value of the subject mostly lies in the very extensive field of practice which it opens in the elements of geometry and the operations of algebra. 2. Trigonometry: the subject of the present treatise; on which I proceed to speak more at length.

The notion which is now studied under the name of Trigonometry, is that of magnitude in a state of alternating increase and decrease, or *periodic magnitude*. The term itself merely implies the measurement of triangles, as geometry does that of the earth; and it is still convenient to refer the measurement of triangles (and other figures) to trigonometry, but only as a minute and isolated application.

Taking the primary idea of quantity alternately increasing and decreasing, it is obviously of fundamental importance to detect a proper method of measurement. The circle presents itself for the purpose in the following way. Conceiving a periodic change of magnitude to run through its whole cycle in a given time, let a point revolve uniformly round a circle in the same time, starting from the end of a fixed diameter. The height of the point above the diameter is a periodic magnitude, which goes through all its changes in the same time as the given magnitude: and it is, in fact, one of the great objects of trigonometry to express periodic variation whose law is known in any way, by means of the simple species of variation just described.

Upon further examining the question of periodic variation, we discover in geometry, and in geometry only, a species of magnitude which is of necessity periodic, and is utterly exclusive of indefinite increase: namely, *direction.* In speaking of the direction of a line as a magnitude, we mean to imply that all direction is relative, inasmuch as we only judge the direction of one straight line by comparing it with another. No straight line can increase its difference of direction from that of another indefinitely: after a certain quantity of change, coincidence is reproduced. The connexion of direction with length is found to lead to an extension of the algebra of positive and negative quantities, which gives the same power of interpreting $a+b\sqrt{-1}$ relatively to $a$, as exists already in the case of $+a$ or $-a$ relatively to $a$. This is an application of geometry to algebra; and, though there does exist a point of view in which geometry may appear not absolutely indispensable, and which I have described in Chapter IV., yet it is pretty clear that abstraction must advance considerably before it will be safe to abandon reference to the science of space in presenting algebra complete to the beginner. The progress of algebra, in this respect, is very curious. In its infancy, geometrical interpretation was rendered necessary by want of power in its symbolic language; was abandoned as the latter grew to maturity; and is finally had recourse to again, because symbols are now sufficient to express relations of magnitude which do not yet exist except in regard to space.

This modern application of geometry to algebra is traced in Professor Peacock's Report on Analysis to the British Association, printed in the second volume of their Transactions (A.D. 1834). The account there given of this particular point, as well as the rest of the article, should be read by the student of the higher mathematics with great attention, as being equal, in the elementary point of view, and superior in the historical, to any thing which has yet appeared on the subject. The Report cited has rendered any reference to authorities unnecessary.

I have not inserted any thing on the solution of spherical triangles; the subject being one which, though of primary use in Astronomy, is but little connected with the fundamental part of Trigonometry. I should have added a chapter upon the subject, had I not already published a treatise in the Library of Useful Knowledge, to which I am thereby enabled to refer the reader.

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*University College, London,*
*March* 1, 1837.

## ERRATA IN "ELEMENTS OF TRIGONOMETRY."

Page 7, *Corollaries* 1 and 2, *for* the circles C and C′, *read* the circle C′; *for* P *read* P′.

—— 20, line 8, *for* $\frac{y}{r}+\frac{r}{y}$ *read* $\frac{y}{r}\times\frac{r}{y}$.

—— 20, line 8, *for* $\frac{-x}{r}+\frac{r}{-x}$ *read* $\frac{-x}{r}\times\frac{r}{-x}$.

—— 26, line 2, *for* A″O C″, *read* A‴O C″.

—— 26, line 14, *for* C″N, *read* C″N′.

—— 31, line 11, *for* O, *read* 0.

—— 42, line 12, *strike out* sines of the.

—— 81, line 6, *for* $\frac{\varepsilon^{\theta\sqrt{-1}}-\varepsilon^{-\theta\sqrt{-1}}}{2}$ *read* $\frac{\varepsilon^{\theta\sqrt{-1}}-\varepsilon^{-\theta\sqrt{-1}}}{2\sqrt{-1}}$.

—— 81, line 8 from the end, *for* $\varepsilon^{x}+\varepsilon^{2n\pi\sqrt{-1}}$ *read* $\varepsilon^{x}\times\varepsilon^{2n\pi\sqrt{-1}}$

—— 122, line 20, *for* $y-$ *read* $y^{-n}$.

## ERRATA IN "CONNEXION OF NUMBER AND MAGNITUDE."

Preface, page iv. line 17, *for* that, *read* than.

Page 5, lines 7, 8, and 9 from the end, transpose P and X; and Q and Y.

—— 9, line 9, *for* B, *read* A.

—— 10, line 19, *for* $l$F = $f$M, *read* $l$F = $f$L.

—— 22, line 4 from the end, *for* processes embling, *read* process resembling.

—— 27, line 16, *for* $\frac{m'}{n'}$, *read* $\frac{n'}{m'}$.

—— 34, line 13, *for* are, *read* is.

—— 35, line 3, *for* less, *read* greater.

—— 46, line 13, *for* may be named, *read* is named.

—— 46, last line, *for* $v$(B + K), *read* $w$(B + K).

—— 47, line 1, *for* $v$(Q + Z), *read* $w$(Q + Z).

—— 47, line 2, *for* $v$Q *read* $w$Q, and Z may be as small as we please.

—— 53, last line, *for* B + C + F, *read* B + D + F.

—— 58, line 5 from the end, *for* B, *read* D.

—— 61, lines 2, 4, 6, 7, *for* $m$P, *read* $n$P.

—— 66, line 14, *for* in which, *read* in which we see.

—— 74, line 10, *for* V : L′, *read* $V_1$: L′.

—— 76, line 17, *for* $X_n$, *read* $X_p$.

—— 78, line 10, *for* for, *read* from.

—— 79, line 18, *for* $a^{\frac{1}{4}}v^{\frac{1}{8}}$, *read* $a^{\frac{1}{4}}v^{\frac{1}{6}}$.

—— 80, line 16, *for* 10, *read* 10V.

—— 80, line 20, *for* V$k$X, *read* V$k^{x}$.

# ELEMENTS
## OF
# TRIGONOMETRY.

## CHAPTER I.

### DEFINITIONS AND FUNDAMENTAL FORMULÆ OF TRIGONOMETRY IN THE CASE OF ONE ANGLE.

(1.) Trigonometry originally meant simply the measurement of triangles. It now means measurement generally by means of the properties of triangles, in all cases in which the connexion between sides and angles is concerned, not sides only, nor angles only: together with all consequences of such measurements as are useful in the higher parts of mathematics. If algebraical symbols and operations be adopted, as is now universally the case, it is a branch of the application of algebra to geometry.

(2.) If a line U, taken at pleasure, be called the linear unit, or 1 of length, then $2\,U$ is called 2, $\frac{1}{2}U$ is called $\frac{1}{2}$, and so on. And any line incommensurable with U is denoted by a general symbol such as $a$, where, if the line specified were commensurable with U, $a$ would be a number or fraction, and $a\,U$ would represent the line. Let $a\,U$ still represent the line, where, in the theory, $a$ is a symbol for the ratio of the line to U; and, in practice, a line as near as we please to the incommensurable line is taken, namely, $\frac{m}{n}U$, and $\frac{m}{n}$ is substituted in results instead of $a$. I here suppose the student to have read the preliminary treatise: if not, he must be content with the application of the following proposition.

Two lines being given, A and B, either two whole numbers $m$ and $n$ can be found, so that $B=\frac{m}{n}A$; or so that $\frac{m}{n}A$ shall be as near to B as we please.

The same considerations apply to any other magnitudes; to angles for instance.

(3.) Before proceeding to ascertain how a line may depend upon an angle, or an angle upon a line, it may be useful to shew that we are, by means of geometry, able to determine lines from lines, without consideration of angles, and angles from angles, without consideration of lines. The complete determination of some angles of a figure by means of the rest, whenever that is possible, is contained in Prop. 32 of the first book of Euclid. If $a_1, a_2 \dots\dots a_n$ be the $n$ angles of a rectilinear figure which has no re-entering angles (pointing inwards), and if $\pi$ be the angle made by a line and its continuation, or *twice* a right angle, then

$$a_1 + a_2 + a_3 + \quad \cdots\cdots\cdots \quad + a_n = (n-2)\pi$$

In a triangle $$a_1 + a_2 + a_3 = \pi$$

Hence all the angles of a triangle are known when two are known.

(4.) The determination of lines by means of lines depends mostly upon the two following propositions:

I. 47. If A, B, and C, be the sides of a triangle, and if A and B contain a right angle, then the squares on A and B make together an area equal to the square on C.

Let A, B, and C, contain respectively $a$, $b$, and $c$, of any linear unit. Then (Arithmetic, § 234.), if $a$ be a whole number or fraction, the square on the unit (which call T) is contained $aa$ times in the square on A, or the square on A is $aa\mathrm{T}$. Similarly, the square on B is $bb\mathrm{T}$, that on C is $cc\mathrm{T}$; whence

$$aa\mathrm{T} + bb\mathrm{T} = cc\mathrm{T}$$

or T taken $aa + bb$ times is T taken $cc$ times. Therefore,

$$aa + bb = cc$$

[When* $a$, $b$, and $c$, are either of them incommensurable, this equation no longer exists arithmetically. We shall give the strict developement of this proposition in such a case. Firstly, the ratio of the square on A to the square on U (the linear unit) is that compounded of A : U and A : U. Let A be to U as U to X; then (VI. prop. 16.), the rectangle whose sides are A and X is equal in area to the square on U. Therefore, the squares on A and U are two rectangles with a common altitude A, and bases A and X; consequently (VI. 1),

---

* The parts in brackets are for the student who has read the Introductory Treatise in such a manner as to believe he understands it.

The references are to the *articles* of my treatise on *Arithmetic*, the *pages* of my treatise on *Algebra*, and to the books of Euclid.

$$\text{Square on A} : \text{Square on U} :: A : X$$

But A : X is compounded of A : U and U : X, that is, of A : U and A : U. Now, let the ratio of A : U be represented by $a : 1$, and the compound ratio by $aa : 1$. Then we have, proceeding in a similar way with the other sides,

$$\begin{array}{l} \text{Sq. on A} : \text{sq. on U} :: aa : 1 \\ \text{Sq. on B} : \text{sq. on U} :: bb : 1 \end{array}$$

$$\begin{array}{llll} & \text{Sq. on A} + \text{sq. on B} & : \text{sq. on U} :: & aa + bb : 1 \\ \text{But,} & \text{Sq. on C} & : \text{sq. on U} :: & cc \quad : 1 \end{array}$$

$$\begin{array}{ll} \text{Therefore,} & \text{Sq. on A} + \text{sq. on B} :: \text{sq. on C} :: aa + bb : cc \\ \text{But,} & \text{Sq. on A} + \text{sq. on B} = \text{sq. on C, therefore, } aa + bb = cc \end{array}$$

or let the ratio of A : U have any symbol $a : 1$, &c., and let $aa : 1$ represent the ratio compounded of $a : 1$ and $a : 1$, and let $m + n : 1$ mean the ratio of the sum of two magnitudes to U which are severally to U as $m : 1$ and $n : 1$, agreeably to the definitions in the preliminary Treatise, and we have what should in strictness be written

$$aa + bb : 1 :: cc : 1$$

When A, B, and C, are commensurable with U, then $a$, $b$, $c$, mean whole numbers or fractions, and the preceding is reduced to $aa + bb = cc$ as before. If $a$, $b$, and $c$, be whole numbers or fractions, such that $a$U, $b$U, $c$U, are very near to A, B, C, then we have $aa + bb = cc$ very nearly; but if $a$, $b$, and $c$, be really symbols of incommensurable ratios, or rather $a : 1$, $b : 1$, $c : 1$, then the preceding proposition can only be interpreted with reference to magnitude; namely, as $aa\mathrm{U} + bb\mathrm{U} = cc\mathrm{U}$ where $aa\mathrm{U} : \mathrm{U}$ is the ratio compounded of $a\mathrm{U} : \mathrm{U}$ and $a\mathrm{U} : \mathrm{U}$, or A : U and A : U.]

(5.) VI. 4. If A, B, C, and P, Q, R, be the sides of equiangular triangles; namely, the angles opposite to A and P equal, &c. then,

$$A : B :: P : Q, \quad B : C :: Q : R, \quad C : A :: R : P$$

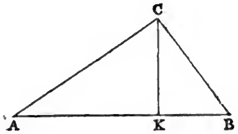
If the ratios of the sides of any triangle be known, the angles are *geometrically* known; that is, we can construct the angles, but cannot yet find their ratios to one another.

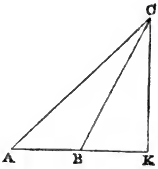
If the sides of the first triangle be $aU$, $bU$, $cU$, and those of the second $pV$, $qV$, $rV$, we have ($a$, $b$, and $c$, being numbers or fractions)

$$\begin{aligned} a : b :: p : q \qquad & aq = pb \\ b : c :: q : r \qquad & br = qc \\ c : a :: r : p \qquad & cp = ra \end{aligned}$$

[If these represent incommensurable ratios, let the student treat these equations in the same manner as $aa + bb = cc$ preceding].

(6.) I now proceed to a case in which lines are numerically determined by lines. Let there be a triangle ABC, in which, U being an arbitrary linear unit, we have





$$\left.\begin{aligned} AB &= cU \\ BC &= aU \\ CA &= bU \end{aligned}\right\}$$ required CK, the perpendicular on $cU$; and AK and KB, the segments[*] of AB.

Let $\quad AK = xU, \qquad KB = yU, \qquad KC = pU$

Then, in the first case, $\quad x + y = c$

...... second .. $\quad x - y = c$

In both $\quad x^2 + p^2 = b^2 \qquad y^2 + p^2 = a^2$

| First Case. | Second Case. |
|---|---|
| $x^2 - y^2 = b^2 - a^2$ | $x^2 - y^2 = b^2 - a^2$ |
| $c(x - y) = b^2 - a^2$ | $c(x + y) = b^2 - a^2$ |
| $c(x + y) = c^2$ | $c(x - y) = c^2$ |
| $2cx = b^2 + c^2 - a^2$ | $2cx = b^2 + c^2 - a^2$ |

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[*] If K be a point in AB, or AB *produced*, AK and KB are called *segments* of AB, *in all cases*. If K lie between A and B, $AK + KB = AB$; if K lie beyond B, $AK - KB = AB$; if K lie beyond A, $BK - KA = AB$.

| First Case. | Second Case. |
|---|---|
| $x = \frac{b^2+c^2-a^2}{2c}$ | $x = \frac{b^2+c^2-a^2}{2c}$ |
| $2cy = c^2+a^2-b^2$ | $2cy = b^2-a^2-c^2$ |
| $y = \frac{c^2+a^2-b^2}{2c}$ | $y = \frac{b^2-a^2-c^2}{2c}$ |

But both cases might have been contained in one, by adopting the conventions of algebra, instead of keeping to arithmetic; for, if we suppose the second case to be made from the first by moving B to the left, we see that BK will be measured in a direction contrary to that which it had at first. If, then, the formulæ of the first case had been used to determine BK in the second case, we should have been warned to measure BK in a direction contrary to that assumed, by its negative sign. For instance, let $c=2$, $a=3$, $b=4$; then we have, taking the formula of the first case,

$$y = \frac{4+9-16}{4} = -\frac{3}{4}$$

That is, BK is $\frac{3}{4}$U in magnitude, not on the same side of K as in the case from whence the formula was arithmetically derived, but on the contrary side. (*Algebra*, chapters I. and II.)

To find $p$, we observe that

$$p^2 = a^2 - y^2 = \frac{4a^2c^2-(c^2+a^2-b^2)^2}{4c^2}$$

$$= \frac{\{2ac-(c^2+a^2-b^2)\}\ \{2ac+c^2+a^2-b^2\}}{4c^2}$$

$$= \frac{\{b^2-(a-c)^2\}\ \{(a+c)^2-b^2\}}{4c^2} = \frac{(a+b+c)(a+c-b)(c+b-a)(b+a-c)}{4c^2}$$

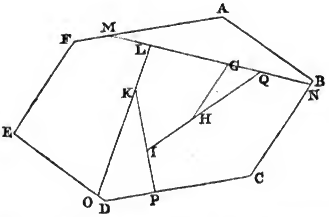
$$\text{Let}\quad \left.\begin{aligned} a+b+c&=2s\\ b+c-a&=2(s-a)\\ c+a-b&=2(s-b)\\ a+b-c&=2(s-c)\end{aligned}\right\}\ \text{then}\ \left\{\begin{aligned} c^2p^2&=4s(s-a)(s-b)(s-c)\\ \\ cp&=2\sqrt{s(s-a)(s-b)(s-c)}\end{aligned}\right.$$

As this result will appear in the sequel, we have chosen it as our instance; and by proceeding with the two propositions in question, it is thus possible to determine lines in terms of lines, without the necessity of employing angles as means of expression. I now pass on to the consideration of an angle connected with a line.

(7.) From the Sixth Book of Euclid it appears, that though given lines will determine angles (as, for instance, the three sides of a triangle being given, the angles can be constructed, both internal and external), yet that it is not necessary to give the lengths; for any other lengths which have the same relative magnitudes will give the same angles. Indeed, the Sixth Book of Euclid is, in great part, an inductive proof of the following proposition. If the absolute lengths of all the lines in a figure be altered in the same ratio, the angles of the figure are not altered: hence, angles depend upon the ratios of lines. The line with which an angle is most connected is the arc of a circle, and it will be necessary to know something more of this figure than can be directly found from the elements. We shall indicate the principal steps necessary, which may be readily filled up by a student who understands the Sixth Book.

(8.) Definition. A bounded figure is called *convex*, when no straight line whatsoever can meet its boundary in more than two points, unless it be itself part of the boundary.

(9.) Theorem. If one convex rectilinear figure be entirely contained within another, the boundary of the contained figure must be less than that of the containing.



Let ABCDEF and GHIKL be the containing and contained figures, then, 1. MN is less than MABN, or MNCDEFM is less than ABCDEFA. 2. Similarly, LODCNL is less than MNCDEFM; 3. KPCNLK is less than LODCNL; 4. KIQLK is less than KPCNLK; 5. KIHGLK is less than KIQLK. Whence the proposition.

(10.) Postulate. Let it be granted that this theorem is also true of convex curvilinear or mixtilinear figures.

(11.) THEOREM. If the homologous sides of two similar rectilinear figures may be made as nearly equal as we please, the figures themselves may be made as nearly equal as we please, both in length of boundary and area. We leave the demonstration to the student, as it is a very simple consequence from the Sixth Book.

(12.) THEOREM. For every rectilinear figure which can be described *in* any one circle, a similar figure can be described *about* any other circle. We shall merely indicate the construction, and leave the student to finish the demonstration. Let C be a circle, and P any inscribed polygon; let C′ be another circle; in it inscribe P′ similar to P. Bisect every side of P′, and draw radii through the points of bisection. From the extremity of each such radius draw a tangent; then will the polygon P″, formed by all the tangents, be similar to P′ and to P.

*Corollary* 1. The circles C and C′ are contained, both as to length of boundary and area, between P and P″

*Corollary* 2. By making the radii sufficiently near to equality, P and P″ may be made (11.), both in boundary length, and area, as near as we please to each other: and if we are at liberty to give P as many sides as we please, and as small (10), the same follows of the circles C and C′ themselves.

*Corollary* 3. And all these results are equally true of sectors of circles which contain the same angle.

(13.) THEOREM. If in a circle a polygon be described, of which no one side exceeds Z in length; then, if Z may be made as small as we please, the polygon and circle may be made as nearly equal as we please, both in boundary length, and area.

For, under these circumstances, as may be easily shewn, the inscribed and similarly superscribed polygons may be made as nearly equal as we please in both respects; and (10.) the circle lies between them in both.

*Corollary.* The same is true of any sector of a circle.

(14.) THEOREM. The circumferences of circles (or of similar sectors) are as their radii, and their areas as the squares on the radii.

Let the radii be R and R′, and describe in the circles two similar polygons, having all their sides severally less than Z. Let P and P′ be the whole boundary lengths of the polygons, and let C and C′ be the circumferences of the circles. But the boundary lengths are proportional to the lengths of two homologous sides, which are to each other as the radii, and (12.) by making Z sufficiently small, these same lengths may be $C-K$ and $C'-K'$, where K and K' are as small as we please. Thence, we have (however small K and K' may be),

$$C-K : C'-K' :: R : R'$$

If possible, let C be to C' not as R to R', but, firstly, in a greater ratio. Then

C is to C' more than $C-K$ is to $C'-K'$;
or $mC$ exceeds $nC'$ where $mC-mK$ does not exceed $nC'-nK'$.
Let $mC=nC'+V$; then $nC'+V-mK$ does not exceed $nC'-nK'$,
and, still more, does not exceed $nC'$

Consequently, $mK$ is at least equal to V; but V is determined without reference to K from C and C', and K may be as small as we please. Therefore, it may be so taken that $mK$ shall be less than V. But if the supposition under trial be true, $mK$ must at least be equal to V; therefore, that supposition is not true, or C is not to C' more than R to R'. If possible, let it be less; then we have, by similar reasoning,

$mC$ is less than $nC'$ where $mC-mK$ is not less than $nC'-nK'$.
Let $mC=nC'-V$; then $nC'-V-mK$ is not less than $nC'-nK'$,
still more then $nC'-V$ is not less than $nC'-nK'$;

or, $nK'$ is at least equal to V. This, as before, shews that the supposition cannot be true. Whence the first part of the theorem follows; and the same may be similarly proved of the sectors.

Let A and A' be the areas of the circles, and Q and Q' the squares on the radii; whence it follows, that $A-L$ and $A'-L'$ being the areas of the similar polygons, (12.) L and L' may be made as small as we please. But (VI. prop. 19.) we must have,

$$A-L : A'-L' :: Q : Q'$$

and, precisely as in the former case, we find that A is to A' neither more nor less than Q is to Q'. Hence, A is to A' as Q to Q'.

Now, let the student shew, that the area of a regular circumscribed polygon of $n$ sides is $\frac{1}{2}nrs\mathrm{T}$ (see next article), $s\mathrm{U}$ being one side: and thence, that $c\mathrm{U}$ and $a\mathrm{T}$ being the circumference and area, we must have $a=\frac{1}{2}cr$.

(15.) Algebraically, let $r$U and $r'$U be the two radii, $c$U and $c'$U the circumferences; then we have,

$$c : c' :: r : r' \quad \text{or} \quad \frac{c}{r} = \frac{c'}{r'}$$

If T be the square on U, the areas $a$T and $a'$T' are as $rr$T and $r'r'$T, or,

$$a : a' :: rr : r'r' \qquad \frac{a}{r^2} = \frac{a'}{r'^2}$$

(16.) The ratios represented by $\frac{c}{r}$ and $\frac{a}{r^2}$ are incommensurable; but we have very nearly

$$\frac{c}{r} = 2 \times \frac{355}{113} \qquad \frac{a}{r^2} = \frac{355}{113}$$

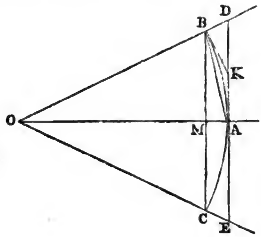
Let the ratio, which is incommensurable, but very near that of 355 : 113, be represented by $\pi$ : 1; or, in the more common way of speaking, which uses incommensurables as commensurables, let $\frac{c}{r}$, which is the same for all circles, be called $2\pi$. Then we have

$$c = 2\pi r \qquad a = \pi r^2$$

$$\pi = \text{nearly } \frac{355}{113} = \text{nearly } 3{\cdot}14159$$

To prove the preceding, we have not yet the means. Let us agree, however, to denote $\frac{c}{r}$, whatever it may be, by $2\pi$.

[But for those students who dislike to leave any thing behind which is afterwards to be proved, we shall establish the following proposition, which assigns the approximate value of $\pi$.



Let BC, DE, be the sides of regular polygons of $n$ sides, inscribed and circumscribed about the circle where radius is OA.

Draw a tangent at B, and complete the figure as shewn. Then will BA, AC, be sides of the polygon of $2n$ sides inscribed, and BK, KA, are halves of sides of the circumscribed polygon of $2n$ sides. Let $I_p$ and $E_p$ mean the areas of the interior and exterior polygons of $p$ sides. Thence, we have,

$$\begin{array}{lllcl}
I_n : I_{2n} & :: & \text{triangle OBC} & : & 2 \times \text{triangle OBA} \\
 & :: & \ldots\ldots \text{ OBM} & : & \ldots\ldots \text{ OBA} \\
 & :: & \text{OM} & : & \text{OA} \\
I_{2n} : E_n & :: & \text{triangle OBA} & : & \text{triangle ODA} \\
 & :: & \text{OB} & : & \text{OD} \\
 & :: & \text{OM} & : & \text{OA}
\end{array}$$

Therefore, $\quad I_n : I_{2n} :: I_{2n} : E_n$

If, then, T be the square on the linear unit U, and if $I_n = i_n T$, &c. we have

$$i_{2n} = \sqrt{i_n e_n}$$

Again (VI. 3.),

$$\begin{array}{rlcl}
\text{AK} : \text{KD} & :: \text{AO} & : & \text{OD} :: \text{MO} : \text{OB} \\
\text{AK} : \text{AD} & :: \text{MO} & : & \text{MO} + \text{OB} \\
 & :: \text{MO} & : & \text{MO} + \text{OA} \\
 & :: I_n & : & I_n + I_{2n}
\end{array}$$

And
$$\begin{array}{rlcl}
E_{2n} : E_n & :: 2 \times \text{triangle AOK} & : & \text{triangle AOD} \\
 & :: 2\text{AK} & : & \text{AD} \\
 & :: 2 I_n & : & I_n + I_{2n}
\end{array}$$

whence
$$e_{2n} = \frac{2\, i_n\, e_n}{i_n + i_{2n}}$$

which formulæ, namely,

$$i_{2n} = \sqrt{i_n e_n} \qquad e_{2n} = \frac{2\, i_{2n}^2}{i_n + i_{2n}}$$

enable us to pass in numbers approximately from the areas of any inscribed or circumscribed polygon, to those of double the number of sides.

Let the linear unit be the radius itself, and T, therefore, the square on the radius. We have, then, on inspection (or IV. 6, 7.)

$$I_4 = 2T \qquad E_4 = 4T$$

$$i_4 = 2 \qquad\qquad e_4 = 4$$

$$i_8 = \sqrt{8} = 2\sqrt{2} \qquad\qquad e_8 = \frac{16}{2+\sqrt{8}} = 8(\sqrt{2}-1)$$

$$i_{16} = 4\sqrt{2-\sqrt{2}} \qquad\qquad e_{16} = \frac{32(2-\sqrt{2})}{2\sqrt{2}+4\sqrt{2-\sqrt{2}}} \text{ \&c.}$$

These expressions shew no very easy law; but if we begin with approximate values of the second pair, namely,

$$i_4 = 2 \qquad\qquad i_8 = 2{\cdot}8284271$$

$$e_4 = 4 \qquad\qquad e_8 = 3{\cdot}3137085$$

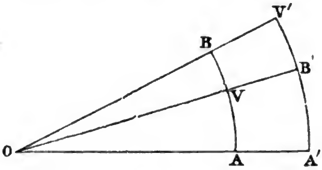
and proceed with these approximations, it will be found, by a calculation which may appear laborious (but, as we have said, this digression is for students whose industry exceeds their disposition to believe without proof), that we have

| Number of sides in the polygon. | Approximate fraction of T contained in the area of the inscribed polygon. | Ditto, Ditto, of the circumscribed polygon. |
|---|---|---|
| 4 | 2·0000000 | 4·0000000 |
| 8 | 2·8284271 | 3·3137085 |
| 16 | 3·0614674 | 3·1825979 |
| 32 | 3·1214451 | 3·1517249 |
| 64 | 3·1365485 | 3·1441184 |
| 128 | 3·1403311 | 3·1422236 |
| 256 | 3·1412772 | 3·1417504 |
| 512 | 3·1415138 | 3·1416321 |
| 1024 | 3·1415729 | 3·1416025 |
| 2048 | 3·1415877 | 3·1415951 |
| 4096 | 3·1415914 | 3·1415933 |
| 8192 | 3·1415923 | 3·1415928 |
| 16384 | 3·1415925 | 3·1415927 |
| 32768 | 3·1415926 | 3·1415926 |

Now, the area of the circle itself always lies between those of the inscribed and circumscribed polygon; and the polygons of 32768 sides, inscribed and circumscribed, do not differ, (slight errors of approximation excepted, which may affect the last place), by the ten-millionth part of a square unit. We may say, then, that the area of the circle is approximately $3 \cdot 14159\,T$, or $\frac{355}{115}T$ very nearly; that is, $\pi = 3 \cdot 14159$ very nearly.]

(17.) *Lemma.* If the same magnitude be $m$ units of one kind, and $m'$ units of another kind, or if $mU = m'U'$, then $m : m' :: U' : U$, and if $U' = kU$, then $m = m'k$.

(18.) Let there be two different angles, BOA, B′OA′, and describe different circles AB, A′B′. Let the given quantities be



the radius OA, and the arc AB; the radius OA′ and the arc A′B′. Required the ratio of the angles.

(VI. 33.) Angle BOA : Angle B′OA′ :: BA : VA

(14.) OA : OA′ :: VA : B′A′

Therefore, BA : B′A′, the ratio of the arcs, is that compounded of the ratios of the angles and the radii. Let U be the linear unit, $\Theta$ the angular unit; and let

$$BA = sU, \quad B'A' = s'U, \quad OA = rU, \quad OA' = r'U,$$

$$\text{Angle } BOA = \theta\,\Theta, \quad \text{Angle } B'OA' = \theta'\,\Theta$$

Then

$$r\theta : r'\theta' :: s : s'$$

or

$$\frac{\theta'}{\theta} = \frac{r}{r'} \cdot \frac{s'}{s}$$

(19.) We have already seen that there has happened a case (*Algebra*, p. 226.) in which one system of suppositions is most convenient in analysis, while another is so in practice; namely, in the choice of a base for logarithms, where 2·7182818 ...... is the base of analysis, and 10 that used in applying logarithms to computations. Just so, in the present instance, there is an angular unit which it is convenient to adopt in investigations, while another unit is universally supposed in practical applications. And the neglect of distinction between these two units is the stumbling-block of the beginner, though the necessity for the distinction is too great to allow us for a moment to think of abandoning one or the other unit.

(20.) 1. The *analytical* or *theoretical* unit (there is no distinct term for it in general use) is the angle which has *an arc equal to the radius*. If, in a solid circular plate, we stretch a thread equal to the radius from point to point of the edge, the thread is then the side of a regular hexagon (IV. 15.) and subtends two-thirds of a right angle. If, then, we bend the thread over the edge, it will (no stretching being supposed) subtend at the centre *somewhat less than two-thirds of a right angle*, which is the first rough notion of the analytical unit.

(21.) In the process of (18.), let $s = r$, and let $\Theta$ be the analytical unit; then $\theta\,\Theta$ is also the same, or $\theta = 1$, and we have

$$\theta' = \frac{s'}{r'}$$

or to determine the number of analytical units in any angle, divide the number of linear units in the arc by that in the radius.

Hence we can easily ascertain how many analytical units there are in one, two, &c. right angles. The radius being $r\,\mathrm{U}$, the whole circumference is $2\pi r\,\mathrm{U}$, and its fourth (or the arc of a right angle) is $\frac{\pi}{2} r\,\mathrm{U}$, the number of units in which is $\frac{\pi}{2} r$. This divided by the number of units in the radius, or $r$, gives $\frac{\pi}{2}$.

| | | |
|---|---|---|
| The right angle in analytical units is | $\frac{\pi}{2}$, | approximately, 1·5707963 |
| Two right angles .......... are | $\pi$ | .......... 3·1415926 |
| Three right angles .............. | $\frac{3}{2}\pi$ | |
| Four right angles .............. | $2\pi$ | |

(22.) 2. The practical method of measuring an angle is well known to be as follows. Let the 90th part of a right angle be called a *degree;* the *sixtieth* part of a degree, a *minute;* the *sixtieth* part of a minute, a second. Let these be denoted by $1^\circ$, $1'$, $1''$, which are not symbols of number, but of magnitude. *They are angles.* We shall always denote the theoretical unit by $\Theta$.

$$1^\circ = 60.1' = 3600.1''$$

A right angle $= 90.1^\circ = 5400.1' = 324000.1'' = 1{\cdot}5707963\ \Theta$ nearly.

Hence $1^\circ = \cdot01745329\ \Theta$ nearly $\qquad \Theta = 206264\cdot8 \,.\, 1''$ nearly

$1' = \cdot000290888\ \Theta \ldots\ldots \qquad \Theta = 3437\cdot747 \,.\, 1' \ldots\ldots$

$1'' = \cdot0000048481\ \Theta \ldots\ldots \qquad \Theta = 57\cdot29578 \,.\, 1^\circ \ldots\ldots$

(23.) It is usual to divide the circumference of a circle into 360 equal parts, and to call each part a degree; the sixtieth part of a degree a minute, &c. To avoid confusion, I shall call these *linear* degrees, minutes, &c. Thus, every circle has its own linear degree; and the greater the circle, the greater the linear degree. On a circle of the earth, the linear degree is 69 miles and a half (a little less) in length. The student will remember that a *linear degree* and *a degree* are two things as different as a line and an angle.

(24.) It is very common among writers on this subject to confound $\pi$ and 180° or 180.1°, $\frac{\pi}{2}$ and 90° or 90.1°. Thus we see sometimes such equations as

$$\pi = 180^\circ \qquad\qquad \frac{\pi}{2} = 90^\circ$$

which are equations of as little title to existence as

$$1760 = 1 \quad \text{or} \quad 20 = 4$$

instead of 1760 yards = 1 mile, or 20 shillings = 4 crowns.

The angle, which in theoretical units is $\pi$, in degrees is 180. But it does not, therefore, follow that $\pi = 180$, any more than that $12 = 4$, because that length which in feet is 12, in yards is 4.

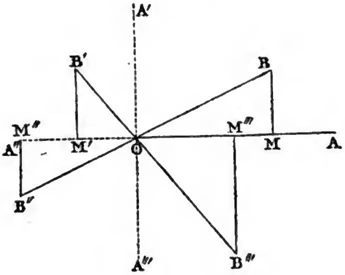
(25.) We now proceed to a more general use of the word *measure*, which frequently occurs in practice. One quantity is said to *measure* another, even when the two are of different kinds, if any change whatever made in the one is accompanied by a proportional change in the other, so that if A of the one give B of the second, $m$A of the one always gives $m$B of the second, whether $m$ be whole or fractional, or the representative of an incommensurable numerical symbol.

Thus, angles are measured by arcs of given circles; for on the same circle any alteration of either arc or subtended angle produces a proportionate alteration in the other. Between given parallels, the areas of rectangles are measured by their bases. But a square is not measured by its side; for if the side be doubled, for instance, the square is not doubled, but quadrupled.

(26.) One magnitude or ratio is determined by another, when, the first being given, the second is given; or at least when, the first being given, the second cannot be any thing we please, but must have one or other of a certain finite number of values. Thus, one angle of a triangle being given (VI. 6.), the ratio of the containing sides determines the other two angles, or rather, determines one of them; for, one angle being given, the sum of the other two is given: the ratio of the containing sides, with the relation just mentioned, determines both the remaining angles.

(27.) If one angle be given (VI. 7.), the ratio of two sides, which do not contain that angle, absolutely determines the other angles, if the given angle be a right angle or more; but if the given angle be less than a right angle, the ratio of the two sides (which do not contain it) determines only two values of each angle, one of which it must be. We shall afterwards have to return to this point.

(28.) An angle, in Euclid, is only one of the angles or openings made by two straight lines, namely, that opening which is less than the opening made by a line and its continuation, or less than two right angles. But any two lines which terminate in their point of meeting, make two angles, one less and one greater than two right angles, the sum of both being four right angles.



(29.) The foundations of trigonometrical notation are as follows: Let a straight line OB, setting out from the position OA, revolve round the point O. Let AA'' and A'A''' be at right angles, and let lines measured from O towards A, or from O towards A', be positive, while lines measured from O towards A'', or from O towards A''', are negative. Let positive angles be described when the revolution makes

OB proceed from OA to OA′, and let negative angles be described when the revolution makes OB proceed from OA to OA‴. And let ratios be positive when both their terms have the same sign, and negative when both their terms have different signs.

Let the Euclidean angle AOB be $\theta\Theta$, where $\Theta$ is the analytical unit, so that $\theta$ is the algebraical symbol for the angle expressed in analytical units; then the whole revolution, or 4 right angles, being $2\pi\Theta$, the angle of two revolutions, or eight right angles, being $4\pi\Theta$, and so on, the line OB is said to make, with OA, an angle

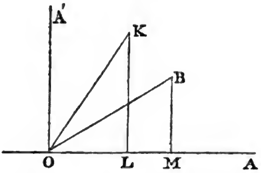
$$\theta\Theta, \quad \text{or} \quad (2\pi+\theta)\Theta, \quad \text{or} \quad (4\pi+\theta)\Theta, \quad \text{\&c.}$$

according as we consider OB to be in its first, second, third, &c. revolution. Or the method by which we take into account the necessity of considering the following proposition, "If a straight line revolve round a point continually, it will, in every succeeding revolution, pass again through all the positions which it had in the first," is by making the following assertion: The angles $\theta$, $2\pi+\theta$, $4\pi+\theta$, &c., and, generally, $2n\pi+\theta$, $n$ being a whole number, are the angles made by OB and OA in the first, second, third, &c., and, generally, in the $(n+1)$th revolution. The angles which it will be necessary to consider as denoting distinct positions, are those which are less than four right angles or $2\pi\Theta$, if we only consider positive angles; or those which are less than two right angles in magnitude, if we consider positive and negative angles; that is, which lie between $+\pi\Theta$ and $-\pi\Theta$. We shall, for the present, confine ourselves to these angles.

(30.) We now proceed to define what we shall call the *primary trigonometrical functions* of an angle, the names of which are the *sine, cosine, tangent, cotangent, secant, cosecant, versed sine*, and *coversed sine* of the angle. And, firstly, whereas in old books on trigonometry these functions are *lines*, in every modern system they are, or should be, defined to be *numbers*, or, in the widest sense, *ratios*. Secondly, the term *sine* is probably from an Arabic word, but the meaning is not well known; *tangent* and *secant* are of obvious Latin derivation; but the definition we shall give has nothing to do with their etymology.

Consider cosine as the abbreviation of "sine of the complement."
...... cotangent .............. "tangent of the complement."
...... cosecant .............. "secant of the complement."

where by the complement of an angle is meant the algebraical excess of a right angle over the angle in question. Thus, $\theta\,\Theta$ being an angle, $\frac{\pi}{2}\Theta-\theta\Theta$ or $\left(\frac{\pi}{2}-\theta\right)\Theta$ is the complement, and $\frac{\pi}{2}-\theta$ its algebraical representation. If the angle be greater than a right angle, the complement is negative. Similarly, if $A\,1^\circ$ be the angle, A being a number or fraction, $(90-A)\,1^\circ$ is the complement.



First, consider an angle AOB less than a right angle. From any point B draw a perpendicular BM; then the ratio of BM to BO is the *sine* of the angle AOB, which we may denote by $\frac{BM}{BO}$, meaning that if BM and BO be expressed in linear units, then the number of units in BM, divided by that in BO, gives the *number* which is called the sine of AOB.

The cosine, or sine of the complement, is thus deduced: Let the triangles OLK and OMB be equal in all respects, namely, KL to OM, &c.; then AOB and AOK make up a right angle, or AOK is the complement of AOB. Its sine is, therefore, $\frac{KL}{KO}$, or $\frac{OM}{OB}$, which is the *cosine* of AOB.

By the tangent of AOB is meant $\frac{BM}{MO}$

The cotangent of AOB is, therefore, $\frac{KL}{LO}$, or $\frac{MO}{MB}$

By the secant of AOB is meant $\frac{BO}{OM}$

The cosecant of AOB is, therefore, $\frac{KO}{OL}$, or $\frac{BO}{BM}$

By the versed sine of AOB is meant $\frac{BO-OM}{BO}$, or 1 — cosine of AOB

The coversed sine of AOB is, then, $\frac{KO-OL}{KO}$, or 1 — sine of AOB

(31.) When the angle is between one and two right angles, as A O B′, then let the same definitions be adopted, as follows (p. 15.):

| | sine. | cosine. | tangent. | cotangent. | secant. | cosecant. |
|---|---|---|---|---|---|---|
| | $\frac{B'M'}{B'O}$ | $\frac{OM'}{OB'}$ | $\frac{B'M'}{OM'}$ | $\frac{OM'}{B'M'}$ | $\frac{B'O}{OM'}$ | $\frac{B'O}{B'M}$ |
| which is | pos. | neg. | neg. | neg. | neg. | pos. |

Remember that O B and O B′, &c. have no sign. Lines only are considered as having signs + and —, which must be in one of two opposite directions.

When the angle is between two and three right angles, as A O B″ (positively measured), then we have for the

| sine. | cosine. | tangent. | cotangent. | secant. | cosecant. |
|---|---|---|---|---|---|
| $\frac{B''M''}{B''O}$ | $\frac{OM''}{OB''}$ | $\frac{B''M''}{OM''}$ | $\frac{OM''}{B''M''}$ | $\frac{B''O}{OM''}$ | $\frac{B''O}{B''M}$ |
| neg. | neg. | pos. | pos. | neg. | neg. |

When the angle is between three and four right angles, as A O B‴ (positively measured), then we have for the

| sine. | cosine. | tangent. | cotangent. | secant. | cosecant. |
|---|---|---|---|---|---|
| $\frac{B'''M'''}{B'''O}$ | $\frac{OM'''}{OB'''}$ | $\frac{B'''M'''}{OM'''}$ | $\frac{OM'''}{B'''M'''}$ | $\frac{B'''O}{OM'''}$ | $\frac{B'''O}{B'''M}$ |
| neg. | pos. | neg. | neg. | pos. | neg. |

The student should verify each of these assertions, which we shall proceed to systematise.

(32.) When an angle is less than a right angle, say it is in the first right angle; when between one and two right angles, say it is in the second right angle, &c. Now, remember the following table:

| | | | | |
|---|---|---|---|---|
| sine and cosecant | + | + | — | — |
| cosine and secant | + | — | — | + |
| tangent and cotangent | + | — | + | — |

which will be found, on examination, to contain the results of the preceding articles thus. I wish to know whether the *sine*, in the *third* right angle, be positive or negative; repeat

*sine* and cosecant, plus, plus, *minus*, minus,

the *third* of which is *negative*, the answer required. Examine the preceding results, and see that this table contains them all.

(33.) Now, from the definitions, make it appear that the second column of assertions below is a translation of the first.

| | |
|---|---|
| The side of a right angled triangle cannot exceed the hypothenuse. | No sine or cosine can exceed unity. |
| The hypothenuse of a right angled triangle cannot be less than a side. | No secant or cosecant can be less than unity. |
| Two lines, of any ratio whatsoever, may be the sides of a right angled triangle. | A tangent or cotangent may have any value whatsoever. |

All this is relative to numerical magnitudes, independently of sign.

*Corollary.* Versed sines and coversed sines are always positive, and never exceed 2.

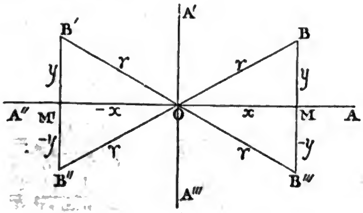
(34.) Now make the following abbreviations.

Let $\theta\,\Theta$ or $A.1^\circ$ be the angle.

| | | | |
|---|---|---|---|
| Let | sine of $\theta\,\Theta$ be written | $\sin\theta$ and read | *sine* $\theta$ |
| | cosine ............ | $\cos\theta$ ...... | *cosine* $\theta$ |
| | tangent ............ | $\tan\theta$ ...... | *tangent* $\theta$ |
| | cotangent ............ | $\cot\theta$ ...... | *cotangent* $\theta$ |
| | secant ............ | $\sec\theta$ ...... | *secant* $\theta$ |
| | cosecant ............ | $\operatorname{cosec}\theta$ ...... | *cosecant* $\theta$ |
| | versed sine ............ | $\operatorname{vers}\theta$ ...... | *versed sine* $\theta$ |
| | coversed sine ............ | $\operatorname{covers}\theta$ ...... | *coversed sine* $\theta$ |

Similarly, let sine of $A.1^\circ$ be written $\sin A$, &c. Observe, that if $\theta\,\Theta$ and $A.1^\circ$ be the same angle, expressed in the two different units, then $\sin\theta = \sin A$, $\cos\theta = \cos A$, &c.; for it is obvious, that the angle itself has the same sine, &c. in whatever units it may be expressed.

(35.) *Fundamental properties implied in the definitions.*



Let U be the linear unit, and let O M and O M′ contain $x$ linear units, the latter being marked $-x$, on account of the contrary direction of O M′. And, first, let A O B (in first right angle) $=\theta\Theta$;

Then

$$\sin\theta \times \operatorname{cosec}\theta = \frac{y}{r} \times \frac{r}{y} = 1$$

$$\cos\theta \times \sec\theta = \frac{x}{r} \times \frac{r}{x} = 1$$

$$\tan\theta \times \cot\theta = \frac{y}{x} \times \frac{x}{y} = 1$$

Let A O B′ (in second right angle) $=\theta\Theta$; then, in the same way,

$$\sin\theta \times \operatorname{cosec}\theta = \frac{y}{r} + \frac{r}{y} = 1 \qquad \cos\theta \times \sec\theta = \frac{-x}{r} + \frac{r}{-x} = 1 \text{ \&c.}$$

so that these formulæ are universal, and we have the following consequences of definition.

The sine and cosecant are reciprocals.
The cosine and secant are reciprocals.
The tangent and cotangent are reciprocals.

We have also

$$\tan\theta = \frac{y}{x} = \frac{y \div r}{x \div r} = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{x}{y} = \frac{x \div r}{y \div r} = \frac{\cos\theta}{\sin\theta}$$

(36.) *Consequences of Euclid 1. 47.* We have the equations (4.)

$$x^2+y^2=r^2,\ x^2+(-y)^2=r^2,\ (-x)^2+y^2=r^2,\ (-x)^2+(-y)^2=r^2$$

The first of which (and the rest may be treated in the same way) may be written in three different forms, namely,

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1 \quad \text{or} \quad \cos^2\theta + \sin^2\theta = 1$$

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2 \quad \text{or} \quad 1 + \tan^2\theta = \sec^2\theta$$

$$1 + \left(\frac{x}{y}\right)^2 = \left(\frac{r}{y}\right)^2 \quad \text{or} \quad 1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

Hence

$$\sin^2\theta = \frac{y^2}{x^2+y^2} = \frac{(y \div x)^2}{1+(y \div x)^2} = \frac{\tan^2\theta}{1+\tan^2\theta}$$

$$\cos^2\theta = \frac{x^2}{x^2+y^2} = \frac{1}{1+(y \div x)^2} = \frac{1}{1+\tan^2\theta}$$

or $$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} \qquad \cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}}$$

Deduce also $$\sin\theta = \frac{1}{\sqrt{1+\cot^2\theta}} \qquad \cos\theta = \frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$$

To these we may add the equations of definition

$$\text{vers}\,\theta = 1 - \cos\theta, \qquad \text{covers}\,\theta = 1 - \sin\theta,$$

(37.) The student should now, as an exercise, express each of the primary functions in terms of every other, as in the following table, where the meaning of $t$ in each column stands at the head, and the value of the expressions in each horizontal line at the beginning.

| | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ | $\cot\theta$ | $\sec\theta$ | $\text{cosec}\,\theta$ |
|---|---|---|---|---|---|---|
| $\sin\theta =$ | $t$ | $\sqrt{1-t^2}$ | $\frac{t}{\sqrt{1+t^2}}$ | $\frac{1}{\sqrt{1+t^2}}$ | $\frac{\sqrt{t^2-1}}{t}$ | $\frac{1}{t}$ |
| $\cos\theta =$ | $\sqrt{1-t^2}$ | $t$ | $\frac{1}{\sqrt{1+t^2}}$ | $\frac{t}{\sqrt{1+t^2}}$ | $\frac{1}{t}$ | $\frac{\sqrt{t^2-1}}{t}$ |
| $\tan\theta =$ | $\frac{t}{\sqrt{1-t^2}}$ | $\frac{\sqrt{1-t^2}}{t}$ | $t$ | $\frac{1}{t}$ | $\sqrt{t^2-1}$ | $\frac{1}{\sqrt{t^2-1}}$ |
| $\cot\theta =$ | $\frac{\sqrt{1-t^2}}{t}$ | $\frac{t}{\sqrt{1-t^2}}$ | $\frac{1}{t}$ | $t$ | $\frac{1}{\sqrt{t^2-1}}$ | $\sqrt{t^2-1}$ |
| $\sec\theta =$ | $\frac{1}{\sqrt{1-t^2}}$ | $\frac{1}{t}$ | $\sqrt{1+t^2}$ | $\frac{\sqrt{1+t^2}}{t}$ | $t$ | $\frac{t}{\sqrt{t^2-1}}$ |
| $\text{cosec}\,\theta =$ | $\frac{1}{t}$ | $\frac{1}{\sqrt{1-t^2}}$ | $\frac{\sqrt{1+t^2}}{t}$ | $\sqrt{1+t^2}$ | $\frac{t}{\sqrt{t^2-1}}$ | $t$ |

Thus, if we would know from the preceding the value of the cosecant in terms of the secant only, we find

$$\text{cosec}\,\theta = \frac{\sec\theta}{\sqrt{\sec^2\theta - 1}}$$

(38.) The next question is to determine what we may call the *transition* values of the primary functions, namely, those which they should have when the angle is *nothing*, or *one*, *two*, or *three* right angles exactly. For, in these cases, a look at the figure of (28.) will shew that the right angled triangle, which is the basis of the definitions, disappears altogether, so that, by any definition yet existing, there are no sines, cosines, &c. But, agreeably to the conventions of algebra, we shall use the following extensions as abbreviations (*Algebra*, p. 156.)

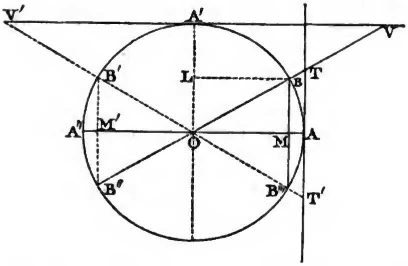
If, when $x$ approaches without limit to $a$, X diminish without limit, let it be said that when $x = a$, $X = 0$: if, in such case, X approach without limit to A, let it be said that when $x = a$, $X = A$: and if X increase without limit, let it be said that when $x = a$ $X = \propto$, or is infinite.

(39.) According to these definitions, and observing what species of variation of magnitude each of the functions undergoes, we have the following table:

| | —<br>0 | ∟<br>1 right angle,<br>or $\frac{\pi}{2}\Theta$ or 90.1° | —⊥—<br>2 right angles,<br>or $\pi\Theta$ or 180.1° | ┌<br>3 right angles,<br>or $\frac{3\pi}{2}\Theta$ or 270.1° |
|---|---|---|---|---|
| sine | 0 | 1 | 0 | —1 |
| cosine | 1 | 0 | —1 | 0 |
| tangent | 0 | ∝ | 0 | ∝ |
| cotangent | ∝ | 0 | ∝ | 0 |
| secant | 1 | ∝ | —1 | ∝ |
| cosecant | ∝ | 1 | ∝ | —1 |
| versed sine | 0 | 1 | 2 | 1 |
| coversed sine | 1 | 0 | 1 | 2 |

The first three lines are the most important. The student may shew that as the angle approaches either of the four intermediate values, its sine, &c. approach the value in the table.

(40.) But the following method will be more clear. We shall determine *linear measures* of the sine, cosine, &c. in the same manner as the arc of a circle is the measure of an angle. Let there be any linear unit U, and with centre O, and radius $OA = U$ describe a circle, and let the angle OAB, or $\theta\Theta$, be that in question. The rest of the figure will need no description. Now, because $\sin\theta = \frac{BM}{BO}$ when BM and BO are expressed in units (let them be $x$U and U) we have $\sin\theta = x$, or $\sin\theta.U = BM$. Consequently, BM and the sine of $\theta$ change in the same proportions; and BM may stand for the sine of $\theta$ (as it might represent a sum of money, an area, or any other magnitude) being a linear measure of it, at the rate, to use a common phrase, of AO to a *unit*. Or let OA be one *measure*, then the fraction of a measure in BM is the sine of the angle AOB, and the fraction of a measure contained in the arc AB, is the fraction of an analytical unit contained in the angle AOB. In precisely a similar manner, OM is a linear measure of the cosine. But when we come to consider the tangent, we see that BM is not a measure of it: for, if the angle increase, the change of the tangent is the combined effect upon the ratio of a simultaneous increase of BM, and decrease of MO. But both are represented in the increase of the line AT; for, by similar triangles, BM : MO :: AT : AO, of which AO remains the same, and AT changes in the same ratio as the tangent. Hence, AT is a linear measure of the tangent of AOB. Similarly, since OM : MB :: A′V : A′O, or AO, A′V is a linear measure of the cotangent; and, since BO : OM :: TO : OA, then TO is a linear measure of the secant. Similarly, OB : BM :: VO : OA′, or VO is a linear measure of the cosecant. Again AM, the linear measure of $1-\cos\theta$ and A′L, that of $1-\sin\theta$, are those of vers $\theta$ and covers $\theta$. Consequently, understanding by the linear unit U the radius OA, and also that the question "How many times?" also implies "What fraction of a time?" and "What times and fraction of a time?" we have the following list of synonymous questions, in which the blanks may be filled up out of the same horizontal line.



| What is the ( ) of the angle AOB? | How many linear units are contained in ( )? |
|---|---|
| sine | BM |
| cosine | OM |
| tangent | AT |

| What is the ( ) of the angle A O B? | How many linear units are contained in ( )? |
|---|---|
| cotangent | A′V |
| secant | O T |
| cosecant | O V |
| versed sine | A M |
| coversed sine | A′L |
| number of analytical units | the arc A B. |

The student may now readily construct the table in (39.) by observing the final states of the linear measures. Thus, as the angle diminishes without limit, the line A′V increases without limit, or (38.) the cotangent of 0 is infinite; and so on.

(41.) (Now, according to the old system of trigonometry, not yet exploded for the beginner, though every person must practically get rid of it before he can advance far in analysis, the line B M is the sine, and not of the angle, but of the arc A B. In this system, there is an infinite number of sines to the same angle, corresponding to all the arcs which that angle can subtend; and, consequently, it is always with reference to the radius supposed that all formulæ must be constructed. For example, it is not true that

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{unless when the radius is 1}$$

but $\sin^2\theta + \cos^2\theta = r^2$ where $r$ is the number of linear units in the radius. This embarrassing consideration is always avoided in practice by making the radius the linear unit, and then substituting for the lines called sines, &c. their numerical proportions to the radius: which amounts in fact to the more modern method).

(42.) We are now to consider, in connection with $\theta$, the angles which exceed or fall short of any whole number of right angles by $\theta$, or which are contained in the following series.

$$\theta - \frac{3\pi}{2} \qquad \theta - \pi \qquad \theta - \frac{\pi}{2} \qquad \theta \qquad \theta + \frac{\pi}{2} \qquad \theta + \pi \text{ \&c.}$$

$$-\frac{3\pi}{2} - \theta \qquad -\pi - \theta \qquad -\frac{\pi}{2} - \theta \qquad -\theta \qquad -\theta + \frac{\pi}{2} \qquad -\theta + \pi \text{ \&c.}$$

all contained in the form $\pm m\frac{\pi}{2} \pm \theta$, where $m$ is a whole number.

But, in the first place, we must observe that any addition or subtraction of four right angles, or multiples of four right angles, produces no change in the position of O B (29.) being, in fact, equivalent to supposing complete revolutions, one or more, to have taken place, leaving OB the same in position as before. The sines, cosines, &c. depending entirely upon the position of OB, and in no way upon the number of revolutions supposed in attaining that position, we must have ($2\pi$ being the numerical symbol of four right angles)

$$\sin\theta = \sin(2\pi+\theta) = \sin(4\pi+\theta) = \sin(6\pi+\theta) \text{ \&c.}$$
$$= \sin(\theta-2\pi) = \sin(\theta-4\pi) = \sin(\theta-6\pi) \text{ \&c.}$$

and generally, F representing the operation by which we pass from an angle to any primary function (*Algebra*, p. 203.) we must have

$$\mathrm{F}(\theta) = \mathrm{F}(\theta+2m\pi)$$

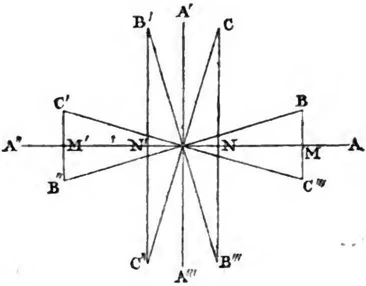
where $m$ is any whole number, positive or negative.

Hence, we may reduce the list in the last page: for, we find that $\mathrm{F}\left(\theta-\frac{\pi}{2}\right)$ is the same as $\mathrm{F}\left(2\pi+\theta-\frac{\pi}{2}\right)$ or $\mathrm{F}\left(\frac{3\pi}{2}+\theta\right)$.

We shall limit ourselves first, to the consideration of the following,

$$\theta \qquad \frac{\pi}{2}\pm\theta \qquad \pi\pm\theta \qquad \frac{3\pi}{2}\pm\theta \qquad 2\pi-\theta$$

omitting $2\pi+\theta$, because its primary functions are those of $\theta$. And, first, let $\theta$ be less than a right angle, so that $m\frac{\pi}{2}+\theta$ must fall in the $(m+1)$th right angle, and $m\frac{\pi}{2}-\theta$ in the $m$th. We have called $\frac{\pi}{2}-\theta$ the complement of $\theta$; it is also usual to call $\pi-\theta$ the supplement of $\theta$; being, when $\theta$ is less than two right angles, the *adjacent* angle of Euclid. Now, draw the following figure, making $\theta$ $\theta$ a small angle for convenience.



O is at the centre (not marked), let the angles AOB, AOC‴, A′OB′, A′OC, A″OB″, A″OC′, A‴OB‴, A″OC″ be all equal to each other, and to $\theta\Theta$. Let the triangles MOB, MOC‴, NOC, NOB‴, N′OB′, N′OC″, M′OC′, M′OB″ be all made equal to each other in every respect; namely, ON to MB, OM to NC, &c. &c. Then we have (all angles being measured positively)

$$\angle\,\mathrm{AOB} = \theta\,\Theta \qquad \angle\,\mathrm{AOB'} = \left(\frac{\pi}{2} + \theta\right)\Theta$$

$$\angle\,\mathrm{AOB''} = (\pi + \theta)\,\Theta \qquad \angle\,\mathrm{AOB'''} = \left(3\frac{\pi}{2} + \theta\right)\Theta$$

$$\angle\,\mathrm{AOC} = \left(\frac{\pi}{2} - \theta\right)\Theta \qquad \angle\,\mathrm{AOC'} = (\pi - \theta)\,\Theta$$

$$\angle\,\mathrm{AOC''} = \left(3\frac{\pi}{2} - \theta\right)\Theta \qquad \angle\,\mathrm{AOC'''} = (2\pi - \theta)\,\Theta$$

From hence we can immediately find any primary function in terms of a primary function of $\theta\Theta$, as follows. Suppose it required to find $\cot\left(3\frac{\pi}{2} - \theta\right)$: we have immediately

$$\cot(\mathrm{AOC''}) = \frac{\mathrm{N'O}}{\mathrm{C''N}} = \frac{\mathrm{BM}\,(\text{with contrary sign})}{\mathrm{OM}\,(\text{with contrary sign})} = +\frac{\mathrm{BM}}{\mathrm{OM}}$$

or

$$\cot\left(3\frac{\pi}{2} - \theta\right) = \tan\theta$$

(43.) Now, in this investigation, there are 42 cases, but they all fall under the following rules for expressing a function of $m\frac{\pi}{2} \pm \theta$ by means of a function of $\theta$. Let $\mathrm{F}\left(m\frac{\pi}{2} + \theta\right)$ be required.

1. *If* m *be odd*, change F into its co-function; namely, sine into cosine, cosine into sine;* tangent into cotangent, cotangent into tangent, &c.: *if* m *be even*, let F remain.

2. Look at the *scale of signs* (32.) of F, namely,

| | | |
|---|---|---|
| for | sine and cosecant | $+ + - -$ |
| | cosine and secant | $+ - - +$ |
| | tangent and cotangent | $+ - + -$ |

and, observing in which right angle $m\frac{\pi}{2} \pm \theta$ falls, prefix the

---

[*] According to our definitions (30.) the co-cosine means the cosine of the complement, or the sine.

sign which answers to the number of that right angle in the scale.

For instance, for $\cot\left(3\frac{\pi}{2}-\theta\right)$, the number of right angles is *odd*, or we write *tan* for *cot*: and $3\frac{\pi}{2}-\theta$ is in the third right angle; $\left(cot\, \overset{+}{1}\ \overset{-}{2}\ \overset{+}{3}\right)$ the proper sign is +

$$\cot\left(3\frac{\pi}{2}-\theta\right) = \tan\theta$$

Let the student go through the cases *from the figure*, and satisfy himself that they agree with this rule.

(44.) The following are some results. The first set is in the definitions.

$$\begin{array}{l|l|l}
\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta & \sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta & \sin(\pi-\theta) = \sin\theta \\
\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta & \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta & \cos(\pi-\theta) = -\cos\theta \\
\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta & \tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta & \tan(\pi-\theta) = -\tan\theta
\end{array}$$

$$\begin{array}{l|l|l}
\sin(\pi+\theta) = -\sin\theta & \sin\left(\frac{3\pi}{2}-\theta\right) = -\cos\theta & \sin\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta \\
\cos(\pi+\theta) = -\cos\theta & \cos\left(\frac{3\pi}{2}-\theta\right) = -\sin\theta & \cos\left(\frac{3\pi}{2}+\theta\right) = \sin\theta \\
\tan(\pi+\theta) = \tan\theta & \tan\left(\frac{3\pi}{2}-\theta\right) = \cot\theta & \tan\left(\frac{3\pi}{2}+\theta\right) = -\cot\theta
\end{array}$$

$$\begin{array}{ll}
\sin(2\pi-\theta) = -\sin\theta & \sin(-\theta) = -\sin\theta \\
\cos(2\pi-\theta) = \cos\theta & \cos(-\theta) = \cos\theta \\
\tan(2\pi-\theta) = -\tan\theta & \tan(-\theta) = -\tan\theta
\end{array}$$

The last set is deduced from that immediately preceding, by subtracting four right angles (42.). They may be deduced immediately, by observing that $MOC''' = -\theta\Theta$ (29.).

To obtain versed and coversed sines, remember that

$$\text{vers}\,\theta = 1-\cos\theta, \qquad \text{covers}\,\theta = 1-\sin\theta$$

Thus, $\text{covers}\left(3\frac{\pi}{2}-\theta\right) = 1-\sin\left(\frac{3\pi}{2}-\theta\right) = 1+\cos\theta$

(45.) When $\theta\Theta$ is greater than a right angle, the results are the same as if it were less than a right angle. An easier demonstration will afterwards apply; in the meantime, suppose $\theta = \pi + \theta'$, and we want, for instance, $\cot\left(3\frac{\pi}{2} - \theta\right)$. We have then,

$$3\frac{\pi}{2} - \theta = \frac{\pi}{2} - \theta' \qquad \cot\left(\frac{3\pi}{2} - \theta\right) = \cot\left(\frac{\pi}{2} - \theta'\right) = \tan\theta'$$

But $$\tan\theta' = \tan(\pi + \theta') = \tan\theta;$$

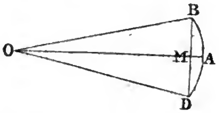
therefore, $$\cot\left(3\frac{\pi}{2} - \theta\right) = \tan\theta$$

the same as before. Let the student work a number of instances of this kind.

By means of the last set of the preceding formulæ, we can easily ascertain $F\left(\theta - m\frac{\pi}{2}\right)$. Suppose $\tan(\theta - \pi)$ is required; then we have

$$\tan(\theta - \pi) = -\tan(\pi - \theta) = -(-\tan\theta) = \tan\theta$$

(46.) We shall now proceed to some theorems connected with the limits of the ratios of trigonometrical functions, (*Alg.* p. 162.).



The ratio of an angle (in analytical units) to its sine, approximates without limit to unity, when the angle is diminished without limit. Let AOB, AOD, be equal angles; then BM = MD, arc AB = arc AD; and, AOB being $\theta\Theta$; or, $\frac{AB}{AO}$ being $\theta$, we have

$$\theta : \sin\theta :: \frac{AB}{AO} : \frac{BM}{AO} :: AB : BM$$
$$:: 2AB : 2BM :: \text{arc}\,BD : \text{chord}\,BD$$

Let the chord BD be the side of an inscribed polygon of $n$ sides; then the greater $n$ is taken, the less does the whole boundary of the polygon (which is $n \times$ chord BD in length) differ from the circumference of the circle (which is $n \times$ arc BD). Let $n \times$ arc BD $=$ $n \times$ chord BD $+ Z$; then can the length Z be made as small as we please, by taking $n$ sufficiently great.

But chord BD $= \frac{\sin\theta}{\theta}$ arc BD, and substitution gives.

$$n \times \text{arc BD} - n \times \frac{\sin\theta}{\theta} \times \text{arc BD} = \text{Z},$$

or
$$\left(1 - \frac{\sin\theta}{\theta}\right) \times n \times \text{arc BD} = \text{Z}$$

that is,
$$\left(1 - \frac{\sin\theta}{\theta}\right) \times \text{circumference} = \text{Z}$$

But as $n$ increases without limit, the angle BOD, and, therefore, BOM, diminishes without limit; and since, in such a case, Z also diminishes without limit, the fraction $1 - \frac{\sin\theta}{\theta}$ diminishes without limit, the circumference being always the same. Hence, $\frac{\sin\theta}{\theta}$ approaches without limit to unity. Let the student try to demonstrate this in the manner of (4.), supposing $\theta$ and $\sin\theta$ to be incommensurable.

The angle of 5 degrees, which is not for any practical purpose a small angle, and which, in analytical units, is ·0872665, has, for its sine, ·0871557; which gives,

$$\frac{\sin(\cdot0872665)}{\cdot0872665} = \frac{\cdot0871557}{\cdot0872665} = \frac{872}{873} \text{ very nearly.}$$

or, when AOB is the eighteenth part of a right angle, if the arc BA were divided into 800 equal parts, BM would be more than 799 of these parts.

(47.) As the angle diminishes without limit, the cosine approaches without limit to unity; and $1-\cos\theta$ diminishes without limit, as also does $\sin\theta$. It will be necessary to examine the ratio of $1-\cos\theta$ to $\sin\theta$ under this change.

$$\frac{1-\cos\theta}{\sin\theta} = \frac{(1-\cos\theta)(1+\cos\theta)}{\sin\theta\ (1+\cos\theta)} = \frac{\sin^2\theta}{\sin\theta(1+\cos\theta)} = \frac{\sin\theta}{1+\cos\theta}$$

of which the numerator diminishes without limit, while the denominator increases with the limit $1+1$ or 2. The fraction, therefore, diminishes without limit, or $\frac{1-\cos\theta}{\sin\theta}$ diminishes without limit.

When $\theta$ is 5 degrees, its sine and cosine are ·0871557 and ·9961947 very nearly. Whence,

$$\frac{1-\cos\theta}{\sin\theta} = \frac{\cdot0038053}{\cdot0871557} < \frac{4}{87}$$

or, for this angle, $1-\cos\theta$ is less than the twentieth part of $\sin\theta$.

If we take 5 minutes instead of 5 degrees (5 minutes in analytical units being ·0014544, its sine the same to seven places of decimals, and its cosine ·9999989), we find, that if $\theta$ were divided into 14000 equal parts, the sine would not be less by so much as one of those parts, and that $1-\cos\theta$ is not the 1300th part of $\sin\theta$.

(48.) Again, since $\frac{\tan\theta}{\sin\theta}=\frac{1}{\cos\theta}$, the limit is unity when $\theta$ diminishes without limit; and since $\frac{\tan\theta}{\theta}=\frac{\tan\theta}{\sin\theta}\cdot\frac{\sin\theta}{\theta}$, the limit of this, in the same circumstance, is $1\times 1$, or 1. And, because

$$\frac{1-\cos\theta}{\theta}=\frac{1-\cos\theta}{\sin\theta}\cdot\frac{\sin\theta}{\theta}$$

therefore also $\frac{1-\cos\theta}{\theta}$ diminishes without limit at the same time as $\theta$.

(49.) We shall now propose, as a problem, the solution of the equation

$$\sin x = \sin y$$

or rather, to find certain solutions; for we have no means as yet of ascertaining that any given number of solutions is the total number. Looking among the results of (44.), we find the following solutions, premising, first, that $x=y$ is one solution: an angle has but one sine.

$$x=y\pm 2\pi \quad x=y\pm 4\pi \quad x=y\pm 6\pi, \text{ \&c.}$$
$$x=\pi-y \quad x=(\pi-y)\pm 2\pi = 3\pi-y \text{ or } -\pi-y$$
$$x=(\pi-y)\pm 4\pi=5\pi-y \text{ or } -3\pi-y, \text{ \&c.}$$

We now propose $\tan mx=\cot ny$. Since $\cot ny=\tan\left(\frac{\pi}{2}-ny\right)$ we have, firstly, $mx=\frac{\pi}{2}-ny$, or $x=\frac{1}{m}\left(\frac{\pi}{2}-ny\right)$. We have, also,

$$mx=\frac{\pi}{2}-ny\pm 2\pi \qquad mx=\frac{\pi}{2}-ny\pm 4\pi, \text{ \&c.}$$

And since $\tan x=\tan(x\pm\pi)$, the following are also solutions:

$$mx=\frac{\pi}{2}-ny\pm\pi \text{ or } \frac{\pi}{2}-ny\pm 3\pi, \text{ \&c.}$$

(50.) The following propositions will be readily proved, especially from the figure in (40.). In the same right angle there are no two sines, or cosines, or tangents, &c., which are equal to each other.

And of angles which do not exceed four right angles, there are two to every sine (or cosecant), and $x$ being one, $\pi - x$ is the other; two to every cosine (or secant), and $x$ being one, $2\pi - x$ is the other; two to every tangent (or cotangent), and $x$ being the lesser, $\pi + x$ is the greater.

Now, $x$ being less than two right angles, so is $\pi - x$; but $\pi + x$ and $2\pi - x$ are greater than two right angles. Consequently, where there is question of the angles of a triangle, the cosine of an angle (or secant), or the tangent (or cotangent), being given, the angle is absolutely determined; for there is but one angle which is contained within the limits of the angles of a triangle (O and $\pi$), to which such cosine, &c. can belong. But, when the sine of an angle is given, or found, as that by which the angle of a triangle is to be determined, there may be two angles within the limits of the problem; for if $x$ be one answer, $\pi - x$ is another.

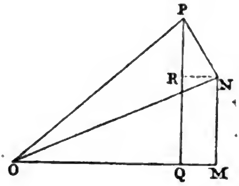
# CHAPTER II.

## FORMULÆ CONNECTED WITH TWO OR MORE ANGLES.

(51.) The doctrine of ratios, as in Euclid, presents the notions which answer to multiplication, division, raising of powers, extraction of roots; every fundamental operation, in fact, *except only addition and subtraction*. The truth is, that it considers two ratios just as two lines are treated in the first book; that is, as subject to the relations of greater, equal, and less, but without any classification or comparison of the various modes of greater and less. From the definitions, it appears that the ratio of $X + Y$ to $Z$ is more than the ratio of $X$ to $Z$; and here the Fifth Book stops. But ratio is a magnitude; we apply the words greater, &c. to two ratios. It is true that the definition of ratio looks more like that of a criterion than of a magnitude; but, as we have seen, the word angle is in the same predicament: the definition of opening, or inclination, is only a rough primary conception, while the useful definition of an angle is the criterion which determines the greater or less, or the equality, of two angles. The rough conception of ratio is *relative magnitude;* that notion, by which a spectator who knew nothing about numbers, would decide whether the picture of a known object was in or out of proportion; that notion, by aid of which savages, who have as little idea of numbers as it is possible for a human being to have, comprehend a map as soon as it is shewn to them, and point out the various sites which they know, as soon as they know whereabouts in the map they are for the time, and the direction of north or south upon it. With this notion comes the following: That the relative magnitudes of $X$ and $Y$ to $Z$, make up the relative magnitude of $X + Y$ to $Z$: whence, we subsequently come to the general definition of addition of ratios; namely, that to add the ratios of $A$ to $B$ and $C$ to $D$, reduce both to other ratios having the same consequent, say $X$ to $Z$, and $Y$ to $Z$; then the sum of the preceding ratios is that of $X + Y$ to $Z$.

We have introduced these considerations again, in order to point out the difference between geometry and algebra, which the following question, being the fundamental proposition of the present chapter, will exemplify. Given the primary functions of two angles, of which the sum is less than a right angle, required the primary functions of their sum. The geometrical solution is as follows:

The first angle being less than a right angle, take any straight line OM, and erect MN, so that the ratio of NM to MO shall be the tangent of the first angle; then will NOM be the angle in question. Then draw NP perpendicular to ON of such length that the ratio of PN to NO shall be the given tangent of the second angle; whence PON is the second angle, and MOP is the sum of the angles. Draw PQ perpendicular to OM, then is the ratio of PQ to PO the sine of the sum required, &c. This geometrical construction is a complete solution within the meaning of the terms *geometrical solution*, with regard to which it is matter of definition that lengths are determined, found, or given, when the extreme points are given. But it is not an algebraical solution, of which it is a condition that no magnitude is given, determined, or found, unless its ratio to some given magnitude of the same kind be given, &c. The geometrical solution is the more easy, because it assumes the harder point, and requires only determination of position; the algebraical solution, which requires ratios, carries the geometrical solution further, and demands consequences with which the geometrical solution, by an express definition of exclusion, has nothing to do. And the student will do well to remember this when he comes to read controversy about the relative value of algebraical and geometrical solutions.



The algebraical solution is as follows, without symbolic language. Draw NR parallel to OM. Then, since PQ is made up of PR and NM, the ratio PQ : PO is the sum of PR : PO and NM : PO. But PR : PO is compounded of PR : PN and PN : PO, or by similar triangles, of OM : ON and PN : PO, and being compounded of given ratios, may be expressed by whatever symbol we adopt to signify composition. Similarly, NM : PO is compounded of NM : NO and NO : PO, two given ratios. Under the common meaning of terms in algebra, we may, if all the pairs be commensurable, and if OP stand for the number of linear units in OP, &c. we proceed thus: let NOM $= \theta\,\Theta$, MOP $= \varphi\,\Theta$, then MOP $=$ $(\varphi + \theta)\,\Theta$, and we have

$$\sin(\varphi+\theta) = \frac{\mathrm{PQ}}{\mathrm{PO}} = \frac{\mathrm{PR}}{\mathrm{PO}} + \frac{\mathrm{NM}}{\mathrm{PO}} = \frac{\mathrm{PR}}{\mathrm{PN}}\cdot\frac{\mathrm{PN}}{\mathrm{PO}} + \frac{\mathrm{NM}}{\mathrm{NO}}\cdot\frac{\mathrm{NO}}{\mathrm{PO}}$$

$$= \frac{\mathrm{MO}}{\mathrm{NO}}\cdot\frac{\mathrm{PN}}{\mathrm{PO}} + \frac{\mathrm{NM}}{\mathrm{NO}}\cdot\frac{\mathrm{NO}}{\mathrm{PO}} = \cos\theta\sin\varphi + \sin\theta\cos\varphi$$

If the ratios be incommensurable, we must either, 1. Imagine commensurables very nearly equal to MO, &c. to be substituted, and the real meaning of the equation will then be (meaning by $(a)$ a very near approximation to $a$), $(\cos\theta)(\sin\varphi) + (\sin\theta)(\cos\varphi)$ is very near to $\sin(\varphi+\theta)$; or, 2. adopt the more general ideas of ratio in the preliminary treatise, and interpret the symbols of operation accordingly. Leaving the student to take which course he can, we now proceed, having obtained in every sense of the terms

$$\sin(\varphi+\theta) = \sin\varphi\cos\theta + \cos\varphi\sin\theta$$

similarly $\cos(\varphi+\theta) = \cos\varphi\cos\theta - \sin\varphi\sin\theta$ as follows:

$$\cos(\varphi+\theta) = \frac{\mathrm{OQ}}{\mathrm{OP}} = \frac{\mathrm{OM}-\mathrm{NR}}{\mathrm{OP}} = \frac{\mathrm{OM}}{\mathrm{ON}}\cdot\frac{\mathrm{ON}}{\mathrm{OP}} - \frac{\mathrm{NR}}{\mathrm{NP}}\cdot\frac{\mathrm{NP}}{\mathrm{NO}}\ \&\mathrm{c}.$$

Now, construct a figure in which $\varphi\,\Theta$ and $\theta\,\Theta$ are each less than a right angle, but their sum greater; shew that the process for the sine remains exactly the same, and that in that for the cosine of the sum, which is negative, OM — RN also becomes negative; whence we still have $\cos(\varphi+\theta)$ *with its sign* $=$ (OM — RN) $\div$ OP, and the two formulæ are precisely as before. Shew also that by aid of

$$\cos^2\varphi + \sin^2\varphi = 1 \qquad \cos^2\theta + \sin^2\theta = 1$$

the sum of the squares of the preceding developements is $= 1$.

(52.) Since these formulæ are universally true, independently of all values of the angles, *within a right angle* (as far as we know yet) they will remain true if instead of $\varphi$, we write $\varphi - \theta$. Do this, which gives

$$\sin\varphi = \sin(\varphi-\theta)\cos\theta + \cos(\varphi-\theta)\cdot\sin\theta$$
$$\cos\varphi = \cos(\varphi-\theta)\cos\theta - \sin(\varphi-\theta)\sin\theta$$

Multiply the first by $\cos\theta$, and subtract the second multiplied by $\sin\theta$, remembering that $\cos^2\theta+\sin^2\theta=1$, which gives, transposing the sides,

$$\sin(\varphi-\theta) = \sin\varphi\cos\theta - \cos\varphi\sin\theta$$
$$\cos(\varphi-\theta) = \cos\varphi\cos\theta + \sin\varphi\sin\theta$$

Which proves that the two first formulæ are true when one of the angles is negative. In $\sin(\varphi+\theta)$ write $-\theta$ for $\theta$, and we have

$$\sin(\varphi-\theta) = \sin\varphi\cos(-\theta) + \cos\varphi\sin(-\theta)$$

or (44.) $\qquad = \sin\varphi\cos\theta - \cos\varphi\sin\theta,$ as just proved.

Endeavour to deduce these propositions from an adaptation of the construction in (51.), and verify the value of the sum of the squares as before.

(53.) We now shew that these formulæ remain true whatever may be the magnitude of the angles. We shall take a case, and recommend the student to acquire dexterity in the management of the formulæ, by trying various others. Let us suppose our first angle to be in the third right angle, and our second in the fourth, so that the sum must be in the sixth at least, or in the seventh. Let $\pi+\theta$, and $\frac{3\pi}{2}+\varphi$ be the analytical units in the angles, and $\theta$ and $\varphi$ must therefore be severally less than a right angle. Then the sum is $\frac{5\pi}{2}+\theta+\varphi$, or $2\pi+\frac{\pi}{2}+\theta+\varphi$, and, therefore, its sine is (42.)

$$\sin\left(\frac{\pi}{2}+\overline{\theta+\varphi}\right) \text{ or } \cos(\theta+\varphi) \text{ or } \cos\theta\cdot\cos\varphi - \sin\theta\cdot\sin\varphi \quad .. \text{ (A)}$$

But
$$\sin(\pi+\theta) = -\sin\theta \quad \text{or} \quad \sin\theta = -\sin(\pi+\theta)$$
$$\cos(\pi+\theta) = -\cos\theta \quad .. \quad \cos\theta = -\cos(\pi+\theta)$$
$$\sin\left(\frac{3\pi}{2}+\phi\right) = -\cos\phi \quad .. \quad \cos\phi = -\sin\left(\frac{3\pi}{2}+\phi\right)$$
$$\cos\left(\frac{3\pi}{2}+\phi\right) = \sin\phi \quad .. \quad \sin\phi = \cos\left(\frac{3\pi}{2}+\phi\right)$$

and, by substitution in (A), we have

$$\sin\left(\pi+\theta+3\frac{\pi}{2}+\phi\right) = -\cos(\pi+\theta)\times-\sin\left(\frac{3\pi}{2}+\phi\right) - \left(-\sin\overline{\pi+\theta}\right)\times\cos\left(\frac{3\pi}{2}+\phi\right)$$

$$\sin\left(\pi+\theta+3\frac{\pi}{2}+\phi\right)=\cos(\pi+\theta)\sin\left(\frac{3\pi}{2}+\phi\right)+\sin(\pi+\theta)\cos\left(\frac{3\pi}{2}+\phi\right)$$

let $\pi+\theta=\theta'$, $\frac{3\pi}{2}+\phi=\phi'$, and we have

$\sin(\theta'+\phi')=\cos\theta'\sin\phi'+\sin\theta'\cos\phi'$, the same as before.

We have then as general formulæ, true for all angles, positive and negative,

$$\sin(\phi+\theta)=\sin\phi\cos\theta+\cos\phi\sin\theta \qquad \sin(\phi-\theta)=\sin\phi\cos\theta-\cos\phi\sin\theta$$
$$\cos(\phi+\theta)=\cos\phi\cos\theta-\sin\phi\sin\theta \qquad \cos(\phi-\theta)=\cos\phi\cos\theta+\sin\phi\sin\theta$$

(54.) We can now, with very slight labour, acquire a large number of very useful formulæ so quickly, that previous description will be unnecessary.

$$\begin{aligned}\sin(\phi+\theta)+\sin(\phi-\theta)&=2\sin\phi\cos\theta\\ \sin(\phi+\theta)-\sin(\phi-\theta)&=2\cos\phi\sin\theta\\ \cos(\phi+\theta)+\cos(\phi-\theta)&=2\cos\phi\cos\theta\\ \cos(\phi+\theta)-\cos(\phi-\theta)&=-2\sin\phi\sin\theta\end{aligned}$$

Since these are always true, we may for $\phi$ and $\theta$ write $\frac{1}{2}(\phi+\theta)$ and $\frac{1}{2}(\phi-\theta)$. Do this, which gives

$$\begin{aligned}\sin\phi+\sin\theta&=2\sin\tfrac{1}{2}(\phi+\theta)\cos\tfrac{1}{2}(\phi-\theta)\\ \sin\phi-\sin\theta&=2\cos\tfrac{1}{2}(\phi+\theta)\sin\tfrac{1}{2}(\phi-\theta)\\ \cos\phi+\cos\theta&=2\cos\tfrac{1}{2}(\phi+\theta)\cos\tfrac{1}{2}(\phi-\theta)\\ \cos\phi-\cos\theta&=-2\sin\tfrac{1}{2}(\phi+\theta)\sin\tfrac{1}{2}(\phi-\theta)\end{aligned}$$

$$\frac{\sin\phi-\sin\theta}{\sin\phi+\sin\theta}=\frac{\tan\frac{1}{2}(\phi-\theta)}{\tan\frac{1}{2}(\phi+\theta)} \qquad \frac{\sin\phi+\sin\theta}{\cos\phi+\cos\theta}=\tan\tfrac{1}{2}(\phi+\theta)$$

(55.) Now, from $2\theta=\theta+\theta$, and from $\sin(\theta+\theta)$ &c. deduce

$$\sin 2\theta=2\sin\theta\cos\theta \qquad \sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\begin{aligned}\cos 2\theta&=\cos^2\theta-\sin^2\theta & \cos\theta&=\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\\ &=1-2\sin^2\theta & &=1-2\sin^2\frac{\theta}{2}\\ &=2\cos^2\theta-1 & &=2\cos^2\frac{\theta}{2}-1\end{aligned}$$

$$1+\cos 2\theta=2\cos^2\theta \qquad 1+\cos\theta=2\cos^2\frac{\theta}{2}$$

$$1-\cos 2\theta=2\sin^2\theta \qquad 1-\cos\theta=2\sin^2\frac{\theta}{2}$$

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta} \qquad \tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta}$$

$$\cot^2\theta = \frac{1+\cos 2\theta}{1-\cos 2\theta} \qquad \tan^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \frac{1-\sin\theta}{1+\sin\theta}$$

(56.) $\tan(\phi+\theta) = \dfrac{\sin(\phi+\theta)}{\cos(\phi+\theta)} = \dfrac{\sin\phi.\cos\theta + \cos\phi\sin\theta}{\cos\phi\ \cos\theta - \sin\phi\sin\theta}$

divide both terms of the last fraction by $\cos\phi.\cos\theta$

$$\tan(\phi+\theta) = \frac{\tan\phi+\tan\theta}{1-\tan\phi\tan\theta};$$

similarly, $$\tan(\phi-\theta) = \frac{\tan\phi-\tan\theta}{1+\tan\phi\tan\theta};$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \qquad \tan\theta = \frac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{1}{2}\theta}$$

$$\tan\theta + \tan\phi = \frac{\sin(\theta+\phi)}{\cos\theta\cos\phi} \qquad \tan\theta - \tan\phi = \frac{\sin(\theta-\phi)}{\cos\theta.\cos\phi}$$

(57.) The formulæ in (54.) reduce multiplication to addition in a way which may remind us of logarithms, and we shall see more of the same sort of analogy before we have finished. They give,

$$\begin{aligned}\sin\phi\cos\theta &= \tfrac{1}{2}\sin(\phi+\theta) + \tfrac{1}{2}\sin(\phi-\theta)\\ \cos\phi\sin\theta &= \tfrac{1}{2}\sin(\phi+\theta) - \tfrac{1}{2}\sin(\phi-\theta)\\ \cos\phi\cos\theta &= \tfrac{1}{2}\cos(\phi+\theta) + \tfrac{1}{2}\cos(\phi-\theta)\\ \sin\phi\sin\theta &= \tfrac{1}{2}\cos(\phi-\theta) - \tfrac{1}{2}\cos(\phi+\theta)\end{aligned}$$

Let it be required to reduce the product $\cos m\cos n\cos p$;

$\cos m\ \cos n\ \cos p = \frac{1}{2}\cos(m+n).\cos p + \frac{1}{2}\cos(m-n)\cos p =$ $\frac{1}{4}\cos(m+n+p) + \frac{1}{4}\cos(m+n-p) + \frac{1}{4}\cos(p+m-n) +$ $\frac{1}{4}\cos(p+n-m)$, by applying the same formulæ twice.

(58.) We may apply this method to ascertain the *n*th power of a sine or cosine in terms of the sines and cosines of the multiples of the angle, as follows; by (55.)

$$\cos^2\theta = \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta \qquad \cos^3\theta = \tfrac{1}{2}\cos\theta + \tfrac{1}{2}\cos 2\theta\cos\theta$$
$$= \tfrac{1}{2}\cos\theta + \tfrac{1}{2}(\tfrac{1}{2}\cos 3\theta + \tfrac{1}{2}\cos\theta) \qquad (2\theta+\theta = 3\theta,\ 2\theta-\theta = \theta)$$
$$= \tfrac{3}{4}\cos\theta + \tfrac{1}{4}\cos 3\theta \quad \text{or} \quad 4\cos^3\theta = 3\cos\theta + \cos 3\theta$$

Multiply by $2\cos\theta$ (we thus avoid fractions)

$$\begin{aligned}8\cos^4\theta &= 6\cos^2\theta + 2\cos 3\theta\cos\theta\\ &= 3 + 3\cos 2\theta + \cos 4\theta + \cos 2\theta = 3 + 4\cos 2\theta + \cos 4\theta\end{aligned}$$

Proceeding in this way we get the following set of equations:

$$\begin{aligned}
\cos\theta &= \cos\theta \\
2\cos^2\theta &= \cos 2\theta + 1 \\
4\cos^3\theta &= \cos 3\theta + 3\cos\theta \\
8\cos^4\theta &= \cos 4\theta + 4\cos 2\theta + 3 \\
16\cos^5\theta &= \cos 5\theta + 5\cos 3\theta + 10\cos\theta \\
32\cos^6\theta &= \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10 \\
64\cos^7\theta &= \cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta \\
128\cos^8\theta &= \cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35 \\
\&c. &\quad \&c. \qquad \&c. \qquad \&c. \qquad \&c.
\end{aligned}$$

Again, $2\sin^2\theta = 1 - \cos 2\theta$, $4\sin^3\theta = 2\sin\theta - 2\cos 2\theta . \sin\theta$

(54.) $(2\cos 2\theta . \sin\theta = \sin 3\theta - \sin\theta)$ $\quad = 2\sin\theta - \sin 3\theta + \sin\theta$

$= 3\sin\theta - \sin 3\theta$

$$\begin{aligned}
8\sin^4\theta &= 6\sin^2\theta - 2\sin 3\theta . \sin\theta \\
&= 3 - 3\cos 2\theta - (\cos 2\theta - \cos 4\theta) = 3 - 4\cos 2\theta + \cos 4\theta
\end{aligned}$$

Proceeding in this way, we get equations which may be thus most systematically arranged:

$$\begin{aligned}
\sin\theta &= \sin\theta \\
-2\sin^2\theta &= \cos 2\theta - 1 \\
-4\sin^3\theta &= \sin 3\theta - 3\sin\theta \\
8\sin^4\theta &= \cos 4\theta - 4\cos 2\theta + 3 \\
16\sin^5\theta &= \sin 5\theta - 5\sin 3\theta + 10\sin\theta \\
-32\sin^6\theta &= \cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10 \\
-64\sin^7\theta &= \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta \\
128\sin^8\theta &= \cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35 \\
\&c. &\quad \&c. \qquad \&c. \qquad \&c. \qquad \&c.
\end{aligned}$$

Between these two sets there are strong resemblances and strong differences. It appears that the cosine is a much more simple function, in its relations with other cosines, than is the sine in relation to other sines. The alternation of positive and negative signs, *in pairs*, here occurs for the first time. We shall now shew how to form the inverse expressions, namely, $\cos n\theta$, &c. in terms of powers of $\cos\theta$, &c.

(59.) By (55.) we have

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \qquad \sin 2\theta = 2\sin\theta . \cos\theta$$

Let the sine and cosine of $\theta$ be denoted by s and c.

Then $\cos 2\theta = c^2 - s^2 \qquad \sin 2\theta = 2cs$

$$\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta . c - \sin 2\theta . s = (c^2 - s^2)c - 2cs.s = c^3 - 3cs^2$$

$$\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta . c + \cos 2\theta . s = 2cs.c + (c^2 - s^2)s = 3c^2 s - s^3$$

$$\cos 4\theta = \cos 3\theta . c - \sin 3\theta . s = c^4 - 6c^2 s^2 + s^4$$

$$\sin 4\theta = \sin 3\theta . c + \cos 3\theta . s = 4c^3 s - 4cs^3$$

and thus we get the following equations:

$$\begin{array}{ll} \cos\theta = c & \sin\theta = s \\ \cos 2\theta = c^2 - s^2 & \sin 2\theta = 2cs \\ \cos 3\theta = c^3 - 3cs^2 & \sin 3\theta = 3c^2 s - s^3 \\ \cos 4\theta = c^4 - 6c^2 s^2 + s^4 & \sin 4\theta = 4c^3 s - 4cs^3 \\ \cos 5\theta = c^5 - 10c^3 s^2 + 5cs^4 & \sin 5\theta = 5c^4 s - 10c^2 s^3 + s^5 \end{array}$$

and so on; the law of which will be hereafter investigated.

(60.) It is easily proved that

$$(\cos\theta + \sin\theta)^2 = 1 + \sin 2\theta \qquad (\cos\theta - \sin\theta)^2 = 1 - \sin 2\theta$$

$$\cos\theta = \pm \tfrac{1}{2}\sqrt{1 + \sin 2\theta} \pm \tfrac{1}{2}\sqrt{1 - \sin 2\theta}$$

$$\sin\theta = \pm \tfrac{1}{2}\sqrt{1 + \sin 2\theta} \mp \tfrac{1}{2}\sqrt{1 - \sin 2\theta}$$

in which the ambiguity of signs will be afterwards discussed.

Also $\cos\theta = \sqrt{\tfrac{1}{2}(1 + \cos 2\theta)} \qquad \sin\theta = \sqrt{\tfrac{1}{2}(1 - \cos 2\theta)}$

(61.) Multiply together the fifth and sixth in (54.), and obtain

$$\sin(\varphi + \theta)\sin(\varphi - \theta) = \sin^2\varphi - \sin^2\theta$$

(62.) We shall now proceed to some cases, in which the sines, &c. may be exhibited numerically. But, first, by means of this theorem, namely, that $(m^2 + n^2)U$, $2mnU$, and $(m^2 - n^2)U$, are the sides of a right angled triangle, U being any linear unit, we can at pleasure find the means of verifying the preceding formulæ in the most exact manner.

For, if $\sin\theta = \dfrac{2mn}{m^2 + n^2}$, then $\cos\theta = \dfrac{m^2 - n^2}{m^2 + n^2} \quad \tan\theta = \dfrac{2mn}{m^2 - n^2}$

Let $m = 2$, $n = 1$; then $\sin\theta = \dfrac{4}{5}$, $\cos\theta = \dfrac{3}{5}$, $\tan\theta = \dfrac{4}{3}$, $\sin 2\theta = \dfrac{24}{25}$, $\cos 2\theta = -\dfrac{7}{25}$, $\sin 3\theta = \dfrac{44}{125}$, $\cos 3\theta = -\dfrac{117}{125}$, &c. In such a case as this we do not know the *angle* in question; but we will shew that this rude method, though the labour would be very considerable, is, in theory, an unfailing means of finding the angle to a given sine or cosine, as nearly as we please. Suppose an angle to have a sine and cosine, both positive; that is, to be less than a right angle, or in the first right angle. By finding $\sin\theta$, $\sin 2\theta$, &c., and $\cos\theta$, $\cos 2\theta$, &c., we are able to find $\tan\theta$, $\tan 2\theta$, &c. Now, since the angle in question is less than a right angle, there will be multiples of it in every right angle (one at least); that is, no right angle can be left out, or there must be values of $n\theta$ lying between $m$ and $m+1$ right angles. Consequently, since the tangent in the several right angles is alternately positive and negative, we shall always be warned of the value of the multiple angles passing a whole number of right angles, by a change of sign. The first negative sign will indicate that the multiple has become greater than a right angle; the next change, namely, from negative to positive, that the multiple now exceeds two right angles; and, generally, the $m$th change of sign shews that the multiple in which it appears lies between $m$ and $m+1$ right angles. If, then, we wish to know the angle which belongs to the given tangent within, say one $v$th part of a right angle, we proceed step by step, and find within what right angles $v\theta$ lies, by noting the number of changes of sign in the series $\tan\theta$, $\tan 2\theta$, $\tan 3\theta$ ...... $\tan v\theta$. Let it be between $m$ and $m+1$ right angles; then $\theta$ lies between $\frac{m}{v}$ and $\frac{m+1}{v}$ of a right angle, or is known within one $v$th part of a right angle.

The preceding process would be too long and laborious for practical purposes; but it shews us, theoretically, that *the determination of the angle which has a given primary function, to any degree of nearness, is within the means of common algebra.*

(63.) Coming now to the determination of some primary functions, we shall express the angles both in analytical and practical units. By (34.), all that we have proved of the primary functions of angles represented in the former way, is true of the latter; except only the theorems in (46, &c.), where the angle enters directly with its primary functions. For instance, though 10″ is a very small angle, it is obviously neither proved in (64.), nor true, that $\sin 10'' = 10$ nearly. If we now represent $n$ degrees by $n^\circ$, $n$ minutes by $n'$, &c., we have the following equations:

$$\pi\Theta = 180°,\ \frac{\pi}{2}\Theta = 90°,\ \frac{\pi}{4}\Theta = 45°,\ \frac{\pi}{6}\Theta = 30°,\ \frac{\pi}{3}\Theta = 60°,$$

$$\frac{\pi}{12}\Theta = 15°,\ \frac{3\pi}{4}\Theta = 135°,\ \frac{2\pi}{3}\Theta = 120°,\ \text{\&c. \&c.}$$

1. $\frac{\pi}{4}\Theta$ or 45°. The sine and cosine are evidently equal. In (30.) if $OM = MB = U$, then $OB = \sqrt{2}\,U$, and

$$\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2} \qquad \tan\frac{\pi}{4} = \cot\frac{\pi}{4} = 1$$

2. $\frac{\pi}{6}\Theta$ or 30°. A right-angled triangle, having an angle of 30°, is the half of an equilateral triangle. The side opposite to 30° is half the hypothenuse; hence

$$\sin\frac{\pi}{6} = \frac{1}{2} \quad \cos\frac{\pi}{6} = \sqrt{1-\frac{1}{4}} = \frac{1}{2}\sqrt{3} \quad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \quad \cot\frac{\pi}{6} = \sqrt{3}$$

3. $\frac{\pi}{3}\Theta$ or 60°, is the complement of $\frac{\pi}{6}\Theta$ or 30°. Therefore,

$$\cos\frac{\pi}{3} = \frac{1}{2} \quad \sin\frac{\pi}{3} = \frac{1}{2}\sqrt{3} \quad \tan\frac{\pi}{3} = \sqrt{3} \quad \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$$

4. $\frac{\pi}{12}\Theta$ or 15°, is the half of $\frac{\pi}{6}\Theta$ or 30°. Hence, by the formula in (60.)

$$\cos\frac{\pi}{12} \text{ is either } \frac{1}{2}\sqrt{1+\frac{1}{2}} + \frac{1}{2}\sqrt{1-\frac{1}{2}}$$

or the same, made negative; but it must be positive, whence we have

$$\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} \qquad \text{similarly, } \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$= \frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right) \qquad\qquad = \frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$$

5. $\frac{\pi}{24}\Theta$ or 7°.30′ is the half of $\frac{\pi}{12}\Theta$ or 15°. And by (60.)

$$\cos\frac{\pi}{24} = \frac{1}{2}\sqrt{\left(1+\frac{1}{4}\sqrt{6}-\frac{1}{4}\sqrt{2}\right)} + \frac{1}{2}\sqrt{\left(1-\frac{1}{4}\sqrt{6}+\frac{1}{4}\sqrt{2}\right)}$$

$$\sin\frac{\pi}{24} = \frac{1}{2}\sqrt{\left(1+\frac{1}{4}\sqrt{6}-\frac{1}{4}\sqrt{2}\right)} - \frac{1}{2}\sqrt{\left(1-\frac{1}{4}\sqrt{6}+\frac{1}{4}\sqrt{2}\right)}$$

In this way we may successively find the sine and cosine of 3° 45′, 1° 52′ 30″, 56′ 15″, 28′ 7″·5, 14′ 3″·75, 7′ 1″·875, 3′ 30″·9375, 1′ 45″·46875, 52″·734375, the latter angle being ·8789063 of a minute. But since the sines of small angles are nearly in the ratio of the angles themselves, and since the ratio of magnitudes are the same in whatever units they may be expressed, we have very nearly,

As ·8789063 of 1′ : 1′ : : sine of the first : sine of 1′

the fourth term of which can be found from the preceding three. This method of finding the sine of one minute supposes that we do not know how to express the angle of 1′ in analytical units; if, however, we assume the results of (22.), we see that the sine and angle of 1′ are nearly equal, when the angle is expressed in terms of Θ, and, therefore, that sine 1′ = ·000290888, nearly. But *nearly* is a vague term; we must endeavour to find *how* nearly the sines of the preceding are equal. If we look at the figure in (40.), and the postulate in (10.), we see that AT + TB is greater than the arc AB, or that

$$\frac{\mathrm{AT}+\mathrm{TB}}{\mathrm{OA}} \text{ or } \frac{\mathrm{AT}+\mathrm{OT}-\mathrm{OA}}{\mathrm{OA}} \text{ is greater than } \frac{\mathrm{Arc\,AB}}{\mathrm{OA}}$$

that is $\tan\theta + \sec\theta - 1$ is greater than $\theta$

But $\sec\theta$ is greater than 1; therefore, $\tan\theta$ is greater than $\theta$. Consequently, we find that $\theta - \sin\theta$ is less than $\tan\theta - \sin\theta$, or than $\sin\theta\,(1-\cos\theta) \div \cos\theta$, or than $2\sin\theta\sin^2\frac{\theta}{2} \div \cos\theta$; or

$$\theta - \sin\theta \text{ is less than } \frac{2\sin\theta\left(\sin\frac{\theta}{2}\right)^2}{\cos\theta}$$

If, then, we increase the numerator of the preceding, and diminish the denominator, we in both ways increase the fraction; consequently,

$$\frac{2\theta\left(\frac{\theta}{2}\right)^2}{\cos^2\theta} \text{ is greater than } \frac{2\sin\theta\left(\sin\frac{\theta}{2}\right)^2}{\cos\theta}$$

or $\theta - \sin\theta$ is less than $\frac{1}{2}\,\frac{\theta^3}{1-\sin^2\theta}$; diminish the denominator still further by substituting $\theta$ for $\sin\theta$, and we have finally

$$\theta - \sin\theta \text{ is less than } \frac{1}{2}\,\frac{\theta^3}{1-\theta^2}$$

Or ·000290888 — sin·000290888 is less than a fraction very near to

$$\frac{1}{2}\,\frac{(\cdot 00029)^3}{1-(\cdot 00029)^2} \text{ or } \cdot 00000000001 \text{ very nearly.}$$

Hence, to ten places of decimals, the angle and sine of one minute are the same things: that is, we may assume $\sin 1' = \cdot 000290888$. Now, $\cos\theta = 1 - 2\sin^2\frac{\theta}{2} = 1 - 2\left(\frac{\theta}{2}\right)^2$ very nearly, or $1 - \frac{1}{2}\theta^2$. We have, therefore, $\cos 1' = 1 - \frac{1}{2}(\cdot 0002909)^2$ very nearly $= \cdot 999999958$ very nearly. Knowing thus the sine and cosine of one minute, we might calculate those of 2′, 3′ .... 1°, 1° 1′, .... up to 89° 59′, 90°, and by dividing sines by cosines, we might find the tangents. After which, by taking reciprocals, we might find the cotangents, &c. To put this method in practice would increase all necessary difficulties some hundreds of times; but here, as in (62.), we are not pointing out how a table of sines, &c. should be formed, but merely shewing how it may be done, that the student may not go to the tables, as to results of the possibility of arriving at which he has no comprehension whatever. We can, however, make him see that even the labour of this process may be materially lessened.

1. No cosines need be calculated, nor cotangents, nor cosecants; for in sin 1′ we have cosine 89° 59′; in sin 2′ we have cos 89° 58′ &c.; so that a complete table of sines for all minutes of the right angle is also a table of cosines. For a similar reason, a table of tangents is also one of cotangents, &c.

2. Half of the secants and cosecants may be formed by simple addition, when the rest are known, by the formulæ

$$\operatorname{cosec}\theta = \frac{1}{2}\left(\tan\frac{1}{2}\theta + \cot\frac{1}{2}\theta\right) \quad \sec\theta = \frac{1}{2}\left(\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right) + \cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right)$$

which we leave to the student to prove.

(64.) Suppose, however, that our table is calculated for every minute of the right angle, it remains to see how the truth of the calculated results may be verified. It must be observed, that in all mathematical tables, the danger of an error of printing is greater than that of an error of calculation. An error of the former kind is one to which all places of figures are equally subject; one of the latter is only to be feared in the last figures. The method of verifying a doubtful figure is to calculate the function in question by means of any other functions, using one of the formulæ already obtained. Thus, if there be a doubt about sin 16°, as given in the tables, we may remember that it ought to be the same as 2 sin 8° cos 8°, and we may, therefore, double the product of sin 8° and cos 8°, and compare it with what is given for sin 16°. But, as multiplication and division are tedious operations, compared with addition and subtraction, we should, for our present purpose, prefer such formulæ as contain only the latter pair of operations. And such formulæ have received the name of *formulæ of verification*, meaning, that they are peculiarly applicable for that purpose. We shall now deduce a few of the kind. Let A be an angle measured in degrees, &c.

(54.) (63.) $\sin(30^\circ + A) + \sin(30^\circ - A) = 2\sin 30^\circ . \cos A = \cos A$

Again, the sine of $\frac{\pi}{10}\Theta$, or 18°, may be thus expressed. If $5\theta = \frac{\pi}{2}$, we have

$\cos 3\theta = \sin 2\theta$, or (59.) $\cos^3\theta - 3\cos\theta . \sin^2\theta = 2\cos\theta . \sin\theta$

divide by $\cos\theta$, and substitute $1 - \sin^2\theta$ for $\cos^2\theta$,

$1 - 4\sin^2\theta = 2\sin\theta$; or, taking the positive value of $\sin\theta$

$\sin\frac{\pi}{10} = \frac{\sqrt{5} - 1}{4}$ from which we find

$$\cos\frac{\pi}{5} = 1 - 2\sin^2\frac{\pi}{10} = \frac{\sqrt{5} + 1}{4}$$

Hence we find (54.)

$$\sin(18^\circ + A) + \sin(18^\circ - A) = \frac{\sqrt{5} - 1}{2}\cos A$$

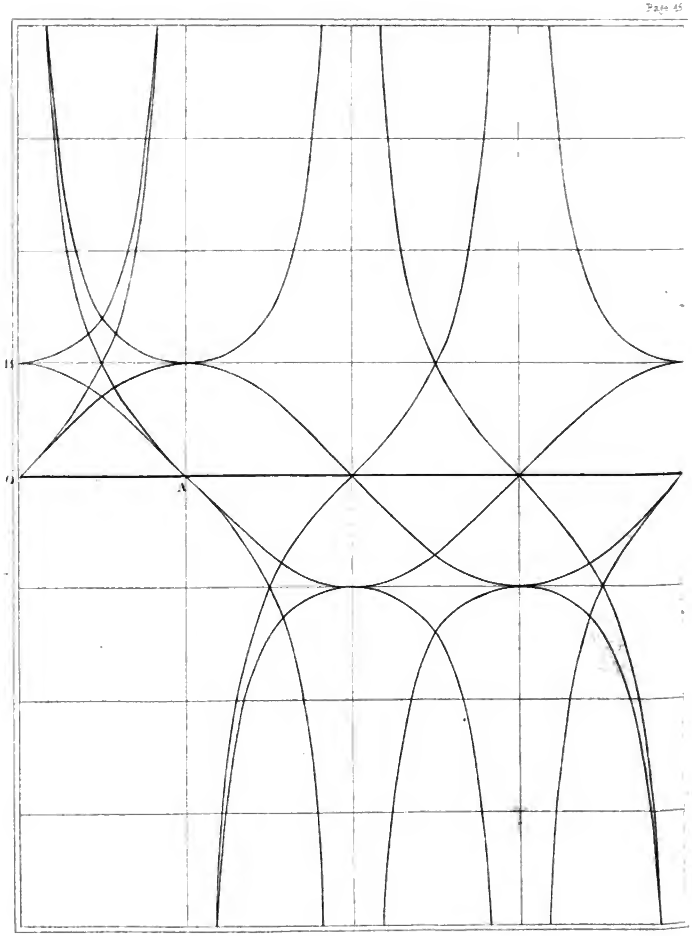
$$\cos(36^\circ + A) + \cos(36^\circ - A) = \frac{\sqrt{5} + 1}{2}\cos A$$

Subtract the first from the second, which gives

$$\cos(36^\circ + A) + \cos(36^\circ - A) = \cos A + \sin(18^\circ + A) + \sin(18 - A)$$

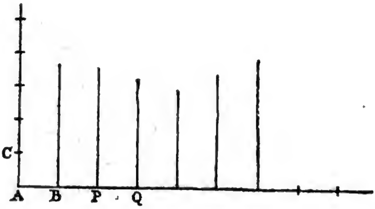
(65.) Before beginning to use the tables, the student should have a good notion of the changes which take place in the magnitudes of the several functions through the first right angle. And he should also take some method of readily remembering the changes of relative magnitude which take place through the four right angles. The best method of doing it is by remembering the general character of the forms of certain curves, which we shall presently proceed to explain, first premising a more simple illustration of the method. Suppose we wished to take a view of the prices of corn (the average per quarter) in different succeeding years.

Page 45



London, Taylor & Walton, Upper Gower Street.

Take two scales perpendicular to each other, as in the figure: let A B represent a year of time, A C a pound sterling of money. If we



begin, say at the year 1790, let B be marked 1790, P 1791, Q 1792, &c. and at each point erect a perpendicular the length of which, A C being twenty shillings, shall represent the price of the quarter of corn for the year. We have thus a better idea of the magnitude of the changes than we could get by looking at a table of prices.

Now, take a similar scale for angles and their primary functions. Let angles be measured on O A, at the rate of a right angle, or 90° to O A; let numbers be measured on the perpendiculars to O A at the rate of O B to a unit. The curves are so drawn that if any angle be laid down on O A (that is, if the proper line be measured from O, which is to O A as the angle in degrees, &c. is to 90°), then the six curves will cut off from the perpendicular six lines which represent (if O B represent 1) the sine, cosine, &c. of the angle. It is now for the student to find out which is the curve of sines, which that of cosines, &c., to examine them attentively, until he perceives the truth of all the theorems in (32.) (39.) &c. and to remember the forms of the curves in such manner that the mere words sine, cosine, &c. shall call up the ideas of the variations of magnitude which are peculiar to the function in question.

For instance, it is long before it is as familiar to a beginner as the word cosine, that the cosine of 0 is 1; the notion always being, that the cosine of *nothing* is *nothing*. A recollection of the manner in which the curve of cosines begins with the angle, would completely remove the liability to this mistake.

It would be one of the most improving exercises which the student could impose upon himself, to draw a considerable number of such curves, provided he can obtain paper ruled* horizontally and vertically at small intervals. It will be quite sufficient to divide a right angle into six equal parts, or to take six intervals of length for the right angle. Four intervals of perpendicular length should represent the unit of the functions in question, by which means that same unit will very nearly represent the *analytical unit* Θ on the line on which angles are measured. Suppose, for instance, the paper is ruled at intervals of a tenth of an inch. Take three inches for a right angle, and two inches for a unit of sine, &c. Suppose the curve required to be that which cuts off $\sin 2\theta - \sin\theta$: then, taking two places of decimals (which will here be sufficient) and remembering that 20 perpendicular subdivisions on the paper count as 1, or each subdivision as ·05, divide the two places by 5, and the result is the number of subdivisions. Also, five subdivisions on the line OA represent 15 degrees, or each subdivision represents 3 degrees. The rest of the process is as follows:

| Angle A | sin A | sin 2 A | sin 2 A — sin A | Subdivisions for the angle. | Subdivisions for sin 2 A — sin A. |
|---|---|---|---|---|---|
| 0 | ·00 | ·00 | ·00 | 0 | 0 |
| 15° | ·26 | ·50 | ·24 | 5 | 4⅘ |
| 30° | ·50 | ·87 | ·37 | 10 | 7⅖ |
| 45° | ·71 | 1·00 | ·29 | 15 | 5⅘ |
| 60° | ·87 | ·87 | ·00 | 20 | 0 |
| 75° | ·97 | ·50 | —·47 | 25 | —9⅖ |
| 90° | 1·00 | ·00 | —1·00 | 30 | —20 |
| 105° | ·97 | —·50 | —1·47 | 35 | —29⅖ |
| 120° | ·87 | —·87 | —1·74 | 40 | —34⅘ |
| 135° | ·71 | —1·00 | —1·71 | 45 | —34⅕ |
| 150° | ·50 | —·87 | —1·37 | 50 | —27⅖ |
| 165° | ·26 | —·50 | —·76 | 55 | —15⅕ |
| 180° | ·00 | ·00 | ·00 | 60 | —0 |

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* The common ruling machine used by stationers will rule paper very well to tenths of inches, with each inch-line broader than the rest. Some years ago, I caused a quantity of paper to be so ruled, which is still on sale with the publisher of this work. In looking at ruled paper, the eye is too accurate a judge: when parallel lines are ruled close together, a very trifling defect is perfectly visible.

The student should now lay down and continue this curve, and others of the same kind.

(66.) Let $h$ be a small angle, of about the value of $1'$. It has been shewn that for more than eight places of decimals, $\cos h = 1$, $\sin h = h$, and hence $\tan h = h$. We have then, with quite sufficient exactness for the tables which it is necessary to use (which never exceed seven places of decimals).

$$\sin(\theta + h) = \sin\theta \cos h + \cos\theta \sin h = \sin\theta + \cos\theta . h$$

$$\cos(\theta + h) = \cos\theta \cos h - \sin\theta \sin h = \cos\theta - \sin\theta . h$$

$$(56.)\ \tan(\theta + h) - \tan\theta = \frac{\sin h}{\cos(\theta + h)\cos\theta} = \frac{h}{\cos^2\theta} \text{ very nearly}$$

or

$$\tan(\theta + h) = \tan\theta + \frac{1}{\cos^2\theta} . h$$

We shall leave the following to the student.

$$\cot(\theta + h) = \cot\theta - \frac{1}{\sin^2\theta} . h$$

$$\sec(\theta + h) = \sec\theta + \frac{\sin\theta}{\cos^2\theta} . h$$

$$\operatorname{cosec}(\theta + h) = \operatorname{cosec}\theta - \frac{\cos\theta}{\sin^2\theta} . h$$

$$\operatorname{vers}(\theta + h) = \operatorname{vers}\theta + \sin\theta . h$$

$$\operatorname{covers}(\theta + h) = \operatorname{covers}\theta - \cos\theta . h$$

That is, $F\theta$ representing any primary function of $\theta$, we have

for sin. tan. sec. or vers. $F(\theta + h) = F\theta + Mh$

for cos. cot. cosec. or covers. $F(\theta + h) = F\theta - Mh$

where M is not a function of $h$, but of $\theta$ only. (Remember that functions beginning with *co.* are all decreasing when the angle increases, in the first right angle). Let $h' = \cdot000290888$ the minute expressed in analytical units; and let $h$ be any angle less than $h'$: then we have

$$\left.\begin{array}{l} F(\theta + h') - F\theta = Mh' \\ F(\theta + h) - F\theta = Mh \end{array}\right\} \text{ for functions which increase with the angle}$$

whence

$$F(\theta + h) = F\theta + \frac{h}{h'}\left\{F(\theta + h') - F\theta\right\}$$

But $F(\theta + h') - F\theta$ is the increment of the function, when the angle receives an increment of one minute in value; it is, therefore, immediately found from the tables, in which the values of $F(\theta+h')$ and $F\theta$ are those which follow each other, if the tables be to every minute, as is usual. Let this increment be denoted by Dif. for *difference*, as in the tables, where the subtraction is made in a separate column. And let $h$ contain $s$ seconds and decimals of seconds. Then will $\frac{h}{h'} = \frac{s}{60}$ and we have (if A be the angle $\theta$ Θ in degrees and minutes.

$$F(A+s) = FA + \frac{s}{60} \times \text{Dif.}$$

and in the same way, if $F\theta$ be a function which decreases when $\theta$ increases, we have

$$F(\theta+h) \quad \text{or} \quad F(A+s) = FA - \frac{s}{60} \times \text{Dif.}$$

where Dif. now stands for $FA - F(A+1')$ and is taken from the tables.

The preceding is, in a more exact form, the representation of a notion which may be more easily given. If we ask what is that function of $h$ which increases uniformly when $h$ increases uniformly (the more easy phrase is, which grows at the same rate as long as $h$ grows at the same rate) the answer is, that the function can only be $P + Mh$ where P and M are independent of $h$. In this function, if for $h$ we write successively

$$h, \quad h+t, \quad h+2t, \quad h+3t, \quad \text{\&c.}$$

the values of the function are

$$(P+Mh), \; (P+Mh)+Mt, \; (P+Mh)+2Mt, \; \text{\&c.}$$

and, similarly, the function $P - Mh$ is of the only form which decreases uniformly when $h$ increases uniformly. If, then, there be a function of $h$ which does not increase uniformly when $h$ undergoes considerable changes of value, but which increases very nearly uniformly when $h$ is small, and undergoes small changes; that is, is very nearly equal to some form of $P + Mh$; the consequence is, that we may for very small values of $h$, and very small changes, treat the function as if it were one which increased uniformly.

To illustrate this, I take out of the table the cosine and sine of 6°, 6° 1′, 6° 2′, and 6° 3′, as follows:

| Angle | Sine | Dif. | Cosine | Dif. |
|---|---|---|---|---|
| 6° 0′ | ·1045285 | | ·9945219 | |
| | | ·0002893 | | ·0000305 |
| 6° 1′ | ·1048178 | | ·9944914 | |
| | | ·0002892 | | ·0000305 |
| 6° 2′ | ·1051070 | | ·9944609 | |
| | | ·0002893 | | ·0000306 |
| 6° 3′ | ·1053963 | | ·9944303 | |

Hence, at and near 6° 1′, the sine is a function which increases uniformly at the rate of ·0002893 per minute of angle, and the cosine diminishes at the rate of ·0000305 per minute. This uniformity of increase or decrease, which obtains when the angle changes through successive minutes, will, *à fortiori*, still remain when the angle changes from second to second in the interval between two minutes. That is, we must, to find the *sine* of 60° 1′ 37″·5, *add* to sin 6° 1′ such a part of ·0002893 as 37½ is of 60, and to find cos 6° 1′ 37″·5 we must subtract from cos 6° 1′ such part of ·0000305 as 37½ is of 60. The process may be performed either by common multiplication or division, or in the manner of the rule called *practice* in commercial arithmetic, as follows:

2893
37
20251
8679
107041
1447
6,0)10848,8
1808

sin 6° 1′ = ·1048178
proportional part for 37″·5 = + ·0001808
sin 6° 1′ 37″·5 = ·1049986

| | | |
|---|---|---|
| for 60 | | 2893 |
| for 30 | ½ | 1447 |
| for 6 | ⅕ | 289 |
| for 1 | ⅙ | 48 |
| for ·5 | ½ | 24 |
| for 37·5 | | 1808 |

305
37
2135
915
11285
153
6,0)1143,8
191

cos 6° 1′ = ·9944914
proportional part for 37·5 = − ·0000191
cos 6° 1′ 37″·5 = ·9944723

| | | |
|---|---|---|
| for 60 | | 305 |
| for 30 | ½ | 153 |
| for 6 | ⅕ | 31 |
| for 1 | ⅙ | 5 |
| for ·5 | ½ | 3 |
| for 37·5 | | 192 |

Nothing will secure accuracy to a single unit in the last place of tables, which are, therefore, always carried a unit further than would otherwise be requisite.

The inverse process to the preceding follows immediately from it. Given F A, F(A + $s$), and F(A + 1′), required $s$. Let P be the given value of F(A + $s$); then we have

$$P = FA + \frac{s}{60}\,\text{Dif.} \qquad \text{Dif.} = F(A + 1') - FA$$

or

$$s = \frac{(P - FA)\,60}{\text{Dif.}}$$

But, if F A be a decreasing function, we have

$$P = FA - \frac{s}{60}\,\text{Dif.} \qquad s = \frac{(FA - P)\,60}{\text{Dif.}}$$

Thus, suppose it is required to find the angle which has ·1052111 for its sine, and also that which has ·9945000 for its cosine.

| | | | |
|---|---|---|---|
| P | ·1052111 | P | ·9945000 |
| sin 6° 2′, or F A | ·1051070 | cos 6°, or F A | ·9945219 |
| P − F A | 1041 | F A − P | 219 |
| | 60 | | 60 |
| | 2893)62460(21·6 | | 305)13140(43·1 |
| | 5786 = $s$ | | 1220 |
| | 4600 | | 940 |
| | 2893 | | 915 |
| | 17070 | | 250 |

Angle required 6° 2′ 21″·6    Angle required 6° 2′ 43″·1

(67.) We must proceed exactly in the same manner with the logarithms of the primary functions; and the law of the increase or decrease may be exhibited as follows. If we have (making $\mu = \cdot 43429\ ..$)

$$F(A + s) = FA + \frac{s}{60}\cdot\text{Dif. this gives } \log F(A + s) = \log\left(FA + \frac{s}{60}\cdot\text{Dif.}\right)$$

$$= \log FA + \log\left(1 + \frac{s}{60}\,\frac{\text{Dif.}}{FA}\right) = \log FA + \frac{s}{60}\,\frac{\mu.\text{Dif.}}{FA} \text{ nearly.}$$

(See *Algebra*, pp. 226 and 237.) But by making $s = 60''$, or 1′, we find

$$\log F(A + 1') = \log FA + \frac{\mu.\text{Dif.}}{FA} \quad\text{or}\quad \frac{\mu.\text{Dif.}}{FA} = \log F(A + 1') - \log FA$$

therefore $\mu$.Dif. $\div$ FA is what is found in the column of differences of the logarithms, and may be taken from the tables. We have then

$$\log F(A+s) = \log FA + \frac{s}{60} \times \text{Dif. of log}$$

or we may remember that the reasoning in page 48 applies to any functions which increase continuously, and, therefore, to the logarithms of the primary functions, as well as to the functions themselves.

(68.) It depends, then, upon the amount of the difference between F(A + 1′) and F(A) whether we can pretend very nearly to find the angle which belongs to any intermediate between them. Look, for instance, at the beginning of the table of logarithmic cosines, which by the arrangement of the tables is the end of the table of logarithmic sines. We have for instance,

$$\begin{array}{ll} \text{log. cos } 0^\circ\ 6' = \cdot 9999993 & \\ & \text{Dif.} = \cdot 0000002 \\ \text{log. cos } 0^\circ\ 7' = \cdot 9999991 & \end{array}$$

We cannot, out of this 2 of difference, make different cosines for every second between 6′ and 7′. The log. cosine here is increasing so slowly, that many successive increments of 1″ will not make it shew a difference of a unit in seven places of decimals. We come to 2° 41′ before an angle increased by 1″ has its log. cosine increased by ·0000001. And if log. cosines are to shew tenths of seconds, that is, if the log. cosine is to increase so rapidly that 0″·1 added to the angle shall make a difference in seven places of decimals, the angle must be upwards of 25°. But when we come to tangents of angles very near 90°, we find that the preceding method fails, because the increases of the log. tangent for successive increases of the angle are far from uniform. Consequently, *when the angle to be found is small, avoid expressing it by means of its cosine, if possible; when it is nearly a right angle, avoid its sine and tangent, if possible.* In the case of a tangent which is very great, denoting an angle near 90°, proceed as follows. Let tan A $= a$, $a$ being a considerable number, so that the angle is nearly a right angle. Remember that (56.)

$$\tan(A - 45^\circ) = \frac{\tan A - 1}{\tan A + 1} = \frac{a-1}{a+1}$$

Find, not A, but A — 45°, from this formula, and the difficulty will disappear; for near 45° the increase of the tangent is very nearly uniform, and also that of its logarithm. For instance, I wish to know, with great exactness, the angle whose tangent is 3000. On looking at the tables I see that it is between 89° 58′ and 89° 59′, but on examining the increase of the tangent, I see as follows:

$$\begin{array}{ll}
\tan 89^\circ\, 57' & 1145{\cdot}9 \\
\tan 89^\circ\, 58' & 1718{\cdot}9 \\
\tan 89^\circ\, 59' & 3437{\cdot}7
\end{array}
\qquad
\left.\begin{array}{l}
\text{Dif.} = \phantom{1}573{\cdot}0 \\
\text{Dif.} = 1718{\cdot}8
\end{array}\right\}
\begin{array}{c}\text{not nearly}\\ \text{equal}\end{array}$$

But, the angle wanted diminished by 45° has for its tangent,

$$\begin{array}{lr}
\dfrac{2999}{3001} = & {\cdot}9993336 \\
\tan 44^\circ\, 58' & {\cdot}9988371 \\
\hline
 & 4965 \\
 & 60
\end{array}$$

$$\begin{array}{lll}
\text{Dif.} = 5813\,\big) & 297900\,\big(51{\cdot}3 & \\
 & 29065 & \\
\hline
 & 7250 & \\
 & 5813 & A - 45^\circ = 44^\circ\, 58'\, 51''{\cdot}3 \\
\hline
 & 14370 & A = 89^\circ\, 58'\, 51''{\cdot}3
\end{array}$$

(69.) Suppose $\cos A = a$, $a$ being very near unity, or the angle very small. We have then

$$\sin^2\frac{1}{2}A = \frac{1}{2}(1-a) \qquad \sin\frac{1}{2}A = \sqrt{\frac{1-a}{2}}$$

which may easily be calculated by logarithms; and from $\frac{1}{2}A$, A can be found. Similarly, if we have $\sin A = a$, where $a$ is very near unity, we have

$$1 - a = 1 - \cos(90^\circ - A) = 2\sin^2\left(45^\circ - \frac{1}{2}A\right)$$

or

$$\sin\left(45^\circ - \frac{1}{2}A\right) = \sqrt{\frac{1-a}{2}}$$

and from $45^\circ - \frac{1}{2}A$, A can be found.

(70.) We have seen that the cosine approaches very near to unity when the angle is small; so near that $1 - \cos\theta$ is a small quantity by the side of $\theta$ itself, when $\theta$ is small (48.). But our future purposes will require a theorem which we shall introduce here, to give a further notion of the rate at which $\cos\theta$ approaches unity. If we take an angle $\theta$, less than $\frac{\pi}{2}$, and form the series $\theta$, $\frac{\theta}{2}$, $\frac{\theta}{3}$, &c. we have a series the terms of which diminish without limit. If we then take

$$\cos\theta, \qquad \cos\frac{\theta}{2}, \qquad \cos\frac{\theta}{3}, \qquad \cos\frac{\theta}{4}, \quad \&c.$$

we have a series of terms approximating without limit to unity. Let us now take

$$\cos\theta, \qquad \left(\cos\frac{\theta}{2}\right)^2 \quad \left(\cos\frac{\theta}{3}\right)^3 \quad \left(\cos\frac{\theta}{4}\right)^4 \quad \&c.$$

We know that the powers of a fraction less than unity decrease, and without limit as the exponents increase (*Algebra*, p. 159), that is, if the fraction operated upon remain the same. But here we have in passing from

$$\cos\theta \quad \text{to} \quad \cos\frac{\theta}{n}, \quad \text{increase}$$

$$\text{from} \quad \cos\frac{\theta}{n} \quad \text{to} \quad \left(\cos\frac{\theta}{n}\right)^n \quad \text{decrease}$$

the question is, as $n$ grows greater and greater, which will predominate, the increase or the decrease. Will $\left(\cos\frac{\theta}{n}\right)^n$, by the decrease which takes place in raising the power, tend to a limit less than unity, or may it be brought as near to unity as we please.

To try this, write

$$\left(\cos\frac{\theta}{n}\right)^n \quad \text{in the form} \quad \left(1 - 2\sin^2\frac{\theta}{2n}\right)^n$$

and remember that if $2\left(\sin\frac{\theta}{2n}\right)^2 = 2\mu\left(\frac{\theta}{2n}\right)^2$ then $\mu$ continually approaches to unity as $n$ is increased. Substitute, which gives (*Algebra*, p. 209, and p. 218, for a similar process)

$$\left(\cos\frac{\theta}{n}\right)^n = \left(1-\frac{\mu\theta^2}{2n^2}\right)^n = 1 - n\frac{\mu\theta^2}{2n^2} + n\frac{n-1}{2}\,\frac{\mu^2\theta^2}{4n^4} - \&c.$$

$$= 1 - \frac{\mu\theta^2}{2n} + \frac{1-\frac{1}{n}}{2}\,\frac{\mu^2\theta^2}{4n^2} - \&c.$$

every term of which, except the first, diminishes without limit when $n$ increases without limit; for $\theta$ remains the same, $\mu$ approaches to unity, and $n$ has increase without limit, causing the same in all the denominators. Hence

$$\left(\cos\frac{\theta}{n}\right)^n \text{ has the limit 1, the same as that of } \cos\frac{\theta}{n}$$

(71.) We shall now examine the limit of $\left(\cos\frac{\theta}{n} + k\sin\frac{\theta}{n}\right)^n$ under like circumstances. It is evident that

$$\left(\cos\frac{\theta}{n} + k\sin\frac{\theta}{n}\right)^n = \left(\cos\frac{\theta}{n}\right)^n \left(1 + k\tan\frac{\theta}{n}\right)^n$$

for $$(p+q)^n = p^n\left(1+\frac{p}{q}\right)^n \quad \text{and} \quad \frac{\sin}{\cos} = \tan$$

in which product the first factor, as just shewn, has unity for its limit, and we must examine that of the second factor. If we make $\tan\frac{\theta}{n} = \mu\frac{\theta}{n}$ then (48.), as $n$ increases without limit, the limit of $\mu$ is unity. Substitute and develope by the binomial theorem, which gives (as in *Algebra*, p. 218),

$$\left(1 + k\mu\frac{\theta}{n}\right)^n = 1 + nk\mu\frac{\theta}{n} + n\frac{n-1}{2}k^2\mu^2\frac{\theta^2}{n^2} + n\frac{n-1}{2}\,\frac{n-2}{3}k^3\mu^3\frac{\theta^3}{n^3} \dots$$

$$= 1 + \mu.k\theta + \frac{1-\frac{1}{n}}{2}\mu^2.k^2\theta^2 + \frac{1-\frac{1}{n}}{2}\,\frac{1-\frac{2}{n}}{3}\mu^3.k^3\theta^3 + \dots$$

Take the limit of both sides (limit of $\mu = 1$)

$$\text{Limit of } \left(1 + k\tan\frac{\theta}{n}\right)^n = 1 + k\theta + \frac{k^2\theta^2}{2} + \frac{k^3\theta^3}{2.3} + \dots = \varepsilon^{k\theta}$$

or $$\text{Limit of } \left(\cos\frac{\theta}{n} + k\sin\frac{\theta}{n}\right)^n = \varepsilon^{k\theta}$$

That is the equation $$\left(\cos\frac{\theta}{n} + k\sin\frac{\theta}{n}\right)^n = \varepsilon^{k\theta}$$

1. is never absolutely true; 2. is very nearly true, if $n$ be great; 3. can be brought as near to truth as we please, by making $n$ sufficiently great. Extract the *arithmetical* $n$th root (*Algebra*, p. 110) of both sides, and we have, as nearly as we please

$$\left.\begin{array}{l}\cos\frac{\theta}{n} + k\sin\frac{\theta}{n} = \varepsilon^{k\frac{\theta}{n}} \\ \text{if } n \text{ may be as great as we please.}\end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{l}\cos\omega + k\sin\omega = \varepsilon^{k\omega} \\ \text{if } \omega \text{ may be as small as we please.}\end{array}\right.$$

Observe that this is independent of the value of $k$. In the last form the result is easy to establish, for when $\omega$ is small $\cos\omega + k\sin\omega$ is nearly $1 + k\omega$, which (*Algebra*, p. 187) is by much the greater part of the developement of $\varepsilon^{k\omega}$. This process, therefore, is one more experience of the confidence to be placed in the developements of $(1+x)^n$ and $\varepsilon^x$ (*Algebra*, c. xi, xii.); and also suggests the propriety of examining further the form $\cos\theta + k\sin\theta$. Multiply two such forms together, and we have

$$(\cos\theta + k\sin\theta)(\cos\theta' + k\sin\theta') = \cos\theta\cos\theta' + k^2\sin\theta\sin\theta' + k\sin(\theta+\theta')$$
$$= \cos\theta\cos\theta' - \sin\theta\sin\theta' + (1+k^2)\sin\theta\sin\theta' + k\sin(\theta+\theta')$$
$$= \cos(\theta+\theta') + k\sin(\theta+\theta') + (1+k^2)\sin\theta\sin\theta'$$

or, if this function of $\theta$, $\cos\theta + k\sin\theta$ be denoted by $f\theta$, we have

$$f\theta \times f\theta' = f(\theta+\theta') + (1+k^2)\sin\theta . \sin\theta' \quad \ldots . \ (1)$$

Multiply again by $f\theta''$ or $\cos\theta'' + k\sin\theta''$, which gives

$$f\theta \times f\theta' \times f\theta'' = f(\theta+\theta'+\theta'') + (1+k^2)\{\sin\theta''\sin(\theta+\theta') + \sin\theta . \sin\theta' . f\theta''\}$$

since by (1) $\quad f(\theta+\theta')f\theta'' = f(\theta+\theta'+\theta'') + (1+k^2)\sin(\theta+\theta')\sin\theta''$

By proceeding in this way we see that if we multiply together $f\theta$, $f\theta'$, $f\theta''$, .... we have an equation of the following form:

$$f\theta . f\theta' . f\theta'' \ldots . = f(\theta+\theta'+\theta''+\ldots .) + (1+k^2)\,\mathrm{V}$$

where V is a function of the angles, and of $k$. This factor $1+k^2$, might be made to simplify the expression materially, if there were such a value of $k$ as that $1+k^2$ should be $=0$, but there is evidently no such algebraical value, positive or negative, for $k^2$ is always positive, and $1+k^2$ greater than 1. We shall hereafter see the consequences of this hint, but we shall leave this formula for the present. As far as we have yet proceeded, every thing seems to render it most likely that, if any function of sines and cosines be identical with a function of common algebra, it is of the form $\cos\theta + k\sin\theta$, which, though not found to be such, is very nearly represented by $\varepsilon^{k\theta}$, when $\theta$ is small. To try this supposition, let us (as an experiment)

make $\qquad \cos\theta + k\sin\theta = \varepsilon^{k\theta}$ for *all* values of $\theta$,

merely to see whether the consequences coincide with those already obtained or not. Then, if this equation be universally true, we have, writing $-\theta$ for $\theta$ (44.),

$$\cos\theta - k\sin\theta = \varepsilon^{-k\theta} = \frac{1}{\varepsilon^{k\theta}}. \quad \text{Let } \varepsilon^{k\theta} = x$$

Then $\qquad 2\cos\theta = x + \frac{1}{x} \qquad 2k\sin\theta = x - \frac{1}{x}$

Again, $\qquad \cos n\theta + k\sin n\theta = \varepsilon^{kn\theta} = \left(\varepsilon^{k\theta}\right)^n = x^n$

$$\cos n\theta - k\sin n\theta = \varepsilon^{-kn\theta} = \left(\varepsilon^{-k\theta}\right)^n = \frac{1}{x^n}$$

Whence $\quad 2\cos n\theta = x^n + \frac{1}{x^n} \qquad 2k\sin n\theta = x^n - \frac{1}{x^n}$

(72.) Let us try some properties of the sine and cosine with these supposed values.

$$\cos 2\theta = 2\cos^2\theta - 1 \qquad \text{or } 2\cos 2\theta = (2\cos\theta)^2 - 2.$$

If our preceding equations be correct, we should have,

$$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2;$$ but this is true, therefore this case does not contradict our assumptions.

Again $\quad \sin 2\theta = 2\sin\theta\cos\theta \quad$ or $\quad 2k\sin 2\theta = (2k\sin\theta)(2\cos\theta)$

But $\quad x^2 - \frac{1}{x^2} = \left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)$ this case, therefore, is no contradiction.

Again, $\quad \cos 2\theta = 1 - 2\sin^2\theta \qquad 2k^2\cos 2\theta = 2k^2 - (2k\sin\theta)^2$

But $\quad 2k^2\cos 2\theta = k^2x^2 + \frac{k^2}{x^2}$

$$2k^2 - (2k\sin\theta)^2 = 2k^2 - \left(x - \frac{1}{x}\right)^2 = 2k^2 + 2 - x^2 - \frac{1}{x^2}$$

to equate these two is therefore to make $k^2 + 1 = 0$, which cannot be.

We may next prove that all the equations contained in

$$2\cos n\theta = x^n + \frac{1}{x^n} \qquad 2k\sin n\theta = x^n - \frac{1}{x^n}$$

for all whole values of $n$ from 0 upwards, *must* be true *if* the two first are true; namely,

$$n = 0 \qquad 2 = x^0 + \frac{1}{x^0} \quad 2k\sin 0 = x^0 - \frac{1}{x^0} \text{ which are true}$$

$$n = 1 \quad 2\cos\theta = x + \frac{1}{x} \quad 2k\sin\theta = x - \frac{1}{x} \quad \ldots\ldots \text{(A)}$$

For it is readily shewn that the truth of any two of these equations involves the truth of the next, as follows. Let $a$ and $a+1$, any two successive values of $n$, give true results, that is, assume

$$2\cos a\theta = x^a + \frac{1}{x^a} \qquad 2k\sin a\theta = x^a - \frac{1}{x^a}$$

$$2\cos(a+1)\theta = x^{a+1} + \frac{1}{x^{a+1}} \quad 2k\sin(a+1)\theta = x^{a+1} - \frac{1}{x^{a+1}}$$

Now, $\quad \frac{(a+2)\theta + a\theta}{2} = (a+1)\theta \qquad \frac{(a+2)\theta - a\theta}{2} = \theta$

(54.) $\cos(a+2)\theta + \cos a\theta = 2\cos(a+1)\theta . \cos\theta$

$\sin(a+2)\theta + \sin a\theta = 2\sin(a+1)\theta \cos\theta$

Therefore
$$2\cos(a+2)\theta = 2\cos(a+1)\theta . 2\cos\theta - 2\cos a\theta$$
$$= \left(x^{a+1} + \frac{1}{x^{a+1}}\right)\left(x + \frac{1}{x}\right) - \left(x^a + \frac{1}{x^a}\right)$$
$$= x^{a+2} + \frac{1}{x^{a+2}}$$

And
$$2k\sin(a+2)\theta = 2k . \sin(a+1)\theta . 2\cos\theta - 2k\sin a\theta$$
$$= \left(x^{a+1} - \frac{1}{x^{a+1}}\right)\left(x + \frac{1}{x}\right) - \left(x^a - \frac{1}{x^a}\right)$$
$$= x^{a+2} - \frac{1}{x^{a+2}}$$

From these it appears that the third follows from the two first; the fourth, from the second and third; the fifth, from the third and fourth, with the second, &c.

We now try the equation

$$(\cos\theta + \sin\theta)^2 = 1 + \sin 2\theta$$

or
$$(k . 2\cos\theta + 2k\sin\theta)^2 = 4k^2 + 2k . 2k\sin 2\theta$$

or
$$\left(kx + \frac{k}{x} + x - \frac{1}{x}\right)^2 = 4k^2 + 2k\left(x^2 - \frac{1}{x^2}\right)$$

But
$$\left((k+1)x + \frac{k-1}{x}\right)^2 = (k+1)^2x^2 + 2(k^2-1) + \frac{(k-1)^2}{x^2}$$
$$= 2k\left(x^2 - \frac{1}{x^2}\right) + (k^2+1)\left(x^2 + \frac{1}{x^2}\right) + 2k^2 - 2$$

which becomes identical with $4k^2 + 2k\left(x^2 - \frac{1}{x^2}\right)$ only on the impossible supposition of $1 + k^2 = 0$.

We shall try one more case.

Since $x^n = \cos n\theta + k\sin n\theta$, and $x = \cos\theta + k\sin\theta$, we have,

$$\cos n\theta + k\sin n\theta = (\cos\theta + k\sin\theta)^n = (\mathrm{c} + k\,\mathrm{s})^n$$

or
$$\cos 2\theta + k\sin 2\theta = \mathrm{c}^2 + 2k\mathrm{c}\mathrm{s} + k^2\mathrm{s}^2 = (\mathrm{c}^2 + 2k\mathrm{c}\mathrm{s} - \mathrm{s}^2)$$
$$\cos 3\theta + k\sin 3\theta = \mathrm{c}^3 + 3k\mathrm{c}^2\mathrm{s} + 3k^2\mathrm{c}\mathrm{s}^2 + k^3\mathrm{s}^3$$
$$= (\mathrm{c}^3 + 3k\mathrm{c}^2\mathrm{s} - 3\mathrm{c}\mathrm{s}^2 - k\mathrm{s}^3)$$
$$\cos 4\theta + k\sin 4\theta = \mathrm{c}^4 + 4k\mathrm{c}^3\mathrm{s} + 6k^2\mathrm{c}^2\mathrm{s}^2 + 4k^3\mathrm{c}\mathrm{s}^3 + k^4\mathrm{s}^4$$
$$= (\mathrm{c}^4 + 4k\mathrm{c}^3\mathrm{s} - 6\mathrm{c}^2\mathrm{s}^2 - 4k\mathrm{c}\mathrm{s}^3 + \mathrm{s}^4)$$

The values in parentheses are those already found in (59.), simply multiplying the sines by $k$, and forming $\cos n\theta + k\sin n\theta$, case after case, altering only the order of the terms, so as to put them under those which they resemble in the results of our present method. We see, then, that the results of our present method coincide with those of (59.), by writing $-1$ for $k^2$, $-k$ for $k^3$, $1$ for $k^4$, &c., which are all algebraical consequences of the single assumption, $k^2 = -1$, which gives $k^3 = -k$, $k^4 = -k^2 = 1$, $k^5 = k$, &c. And from all that has preceded, we deduce the following remark, which we have as much reason to suppose generally true, as instances can give.

(73.) The functions $x + \frac{1}{x}$ and $x - \frac{1}{x}$ have properties corresponding in all respects to the trigonometrical functions $2\cos\theta$ and $2k\sin\theta$; and such that, under the following limitations, the properties of the first may be deduced from those of the second. If an equation which is true of the first functions, undergo substitution of the second for the first, then, if the result do not contain $k$ at all, it is absolutely true of the second; but if it contain powers of $k$ it is never true of the second. Nevertheless, it becomes true of the second, if for the set of even powers of $k$, namely, $k^2$, $k^4$, $k^6$, &c. we substitute $-1$, $+1$, $-1$, &c. and for the set of odd powers of $k$, namely, $k$, $k^3$, $k^5$, $k^7$, &c. we substitute $k$, $-k$, $+k$, $-k$, &c.; in which case, the part independent of $k$ on one side is equal to the part independent of $k$ on the other, and the coefficient of $k$ on one side equal to the coefficient of $k$ on the other. And the representative of $x$ is $\cos\theta + k\sin\theta$, and also $\varepsilon^{k\theta}$.

Nevertheless, $2\cos\theta = x + \frac{1}{x}$ is an impossible equation, except only when $x = 1, \cos\theta = 1$. For, whereas $2\cos\theta$ is never greater than 2, $x + \frac{1}{x}$ is never less than 2. For, if $x + \frac{1}{x}$ were less than 2, $x^2 + 1$ would be less than $2x$, or $x^2 - 2x + 1$, a *square*, would be negative. And, in fact, if we solve

$$x + \frac{1}{x} = 2\cos\theta \text{ we find } x = \cos\theta \pm \sqrt{-1}\sin\theta$$

$$x - \frac{1}{x} = 2k\sin\theta \text{ gives } x = k\sin\theta \pm \sqrt{k^2\sin^2\theta + 1}$$

which agrees in form with the preceding only when $k^2 = -1$.

We have thus laid the foundation of the application of a more abstruse analysis to the primary functions of an angle. We shall first consider the application of our formula to the *solution of triangles*, as it is called, that is, the determination of the remaining parts of a triangle, when enough are given to distinguish it from all others.

# CHAPTER III.

## ON THE SOLUTION OF TRIANGLES.

(74.) LET $a$U, $b$U, $c$U, be the sides of a triangle, U being any given linear unit, and let A, B, C, be the opposite angles, expressed in degrees, minutes, and seconds. When one of the angles is a right angle, it is evident that the tables of sines, cosines, &c. are nothing but registers of the proportions of the sides of such a triangle. Knowing, therefore, any one side, and an angle, we look to the table for the proportion of the other side to it, preferring, of course, the logarithm of the proportion for convenience of calculation.

(75.) The best tables in common use are those of Hutton, which may be procured of any bookseller. The arrangement by which the sine of 18°, for instance, is prevented from being again printed as the cosine of 72°, will be better understood by consulting the table (and remarking the description of the functions at the top and bottom of the page, and the reckoning in minutes downwards on the left hand, and upwards on the right) than by any explanation.

But the following point requires some notice. In every mathematical table which contains both positive and negative quantities, there is such a liability to error in taking out the signs, that it is most useful, and almost necessary, to form the table in such a way that all shall have the same sign. Suppose, for example, that the following table was in frequent use.

$$+6,\ +4,\ -3,\ +2,\ -10,\ -1,\ +8,\ -11.$$

Now 12 being greater than any one of these, add 12 to each, which converts the table into

$$+18,\ +16,\ +9,\ +14,\ +2,\ +11,\ +20,\ +1.$$

Every result in this table is to be added; but 12 is to be subtracted whenever the table is used. There is always both an addition and a subtraction; the sign of the table will not be liable to be read wrong, and the correction of the table is uniform—always a subtraction.

The trigonometrical tables consist of—sines and cosines with logarithms always negative—tangents and cotangents with the same sometimes positive and sometimes negative—and secants and cosecants with logarithms always positive. The plan which is followed is to add 10 to every logarithm in the table, without exception: so that,

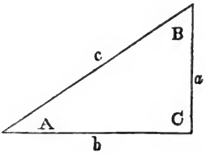
$$\text{True log. FA} = \textit{Tabular}\text{ log. FA} - 10$$
$$\textit{Tabular}\text{ log. FA} = \text{True log. FA} + 10$$

For instance, $\sin 30^\circ = \frac{1}{2}$ $\quad$ log. $\sin 30^\circ = -{\cdot}3010300$.

In the tables we have, $10 - {\cdot}3010300$ or $9{\cdot}6989700$.

(76.) The formulæ for the solution of right-angled triangles, are as follows:

Let C be right angle, *c*U the hypothenuse; then we have,



$$\frac{a}{c} = \sin A = \cos B \qquad a = c\sin A = c\cos B$$

$$\frac{a}{b} = \tan A = \cot B \qquad a = b\tan A = b\cot B$$

$$c^2 = a^2 + b^2 \qquad b = \sqrt{(\overline{c-a}\,.\,\overline{c+a})}$$

But the following formulæ should be remembered in words.

*side* = hypothenuse into sine of *opposite* angle
*side* = hypothenuse into cosine of *adjacent* angle
hypothenuse = *side* by sine of *opposite* angle
hypothenuse = *side* by cosine of *adjacent* angle
*side* = other side into tangent of *opposite* angle
*side* = other side by tangent of *adjacent* angle

(77.) The following are the cases which may occur, and the logarithmic equations for the solution (L sin, &c. mean *tabular* log. sin, &c.; *side*, or *angle*, in Italics, means given side).

1. Given the hypothenuse and a side; required the rest.

$$\text{log. remaining side} = \tfrac{1}{2}\,(\text{log. }\overline{\textit{hyp.} + \textit{side}} + \text{log. }\overline{\textit{hyp.} - \textit{side}})$$
$$\text{L. sin. angle opp. }\textit{side} = 10 + \text{log. }\textit{side} - \text{log. }\textit{hyp.}$$
$$\text{angle opp. other side} = 90^\circ - \text{angle opp. }\textit{side}$$

2. Given the hypothenuse and an angle; required the rest.

$$\begin{aligned}\text{log. side opp. } \textit{angle} &= \text{log. } \textit{hyp.} + \text{L. sin. } \textit{angle} - 10\\ \text{log. side adj. } \textit{angle} &= \text{log. } \textit{hyp.} + \text{L cos. } \textit{angle} - 10\\ \text{other angle} &= 90^\circ - \textit{angle}\end{aligned}$$

3. Given a side and an angle; required the rest.

$$\begin{aligned}\text{Other angle} &= 90^\circ - \textit{angle}\\ \text{log. hyp.} &= 10 + \text{log. } \textit{side} - \text{L sin} \angle \text{ opp. } \textit{side}\\ \text{log. other side} &= \text{log. } \textit{side} + \text{L tan} \angle \text{ adj. } \textit{side} - 10\end{aligned}$$

4. Given the two sides; required the rest.

$$\begin{aligned}\text{L tan. AN ANGLE} &= 10 + \text{log. ITS opp. } \textit{side} - \text{log. other } \textit{side}\\ \text{Other angle} &= 90^\circ - \text{the ANGLE found}\\ \text{log. hyp.} &= 10 + \text{log.} \textit{A SIDE} - \text{L.sin.} \textit{ITS} \text{ opp. angle.}\end{aligned}$$

(78.) There are two cases in which the sine and tangent of an angle are severally to be found; from two equations of these forms

$$\sin A = \frac{a}{c} \qquad \tan A = \frac{a}{b}$$

If in the first case $\frac{a}{c}$ be very near unity (68.), use the equation

$$2\cos^2\left(\frac{90^\circ - A}{2}\right) = \frac{c-a}{c} \quad \text{or} \quad \sin\left(45^\circ - \frac{A}{2}\right) = \sqrt{\frac{c-a}{2c}}$$

If, in the second case, $b$ be very small in comparison of $a$, use

$$\frac{\tan A - 1}{\tan A + 1} = \frac{a-b}{a+b} \quad \text{or} \quad \tan(A - 45^\circ) = \frac{a-b}{a+b}$$

When a side ($b$) is given and a very small adjacent angle (A), the hypothenuse may be determined by its excess above the given side (which is small) as follows:

$$c - b = \frac{b}{\cos A} - b = b\,\frac{(1 - \cos A)}{\cos A} = 2b\sin^2\frac{A}{2} \text{ very nearly.}$$

(79.) Given the hypothenuse and the sum of the two sides; required the rest.

$$(a+b)^2 - c^2 = 2ab = 2c^2 \sin A . \cos A = c^2 \sin 2A$$

$$\text{L.} \sin 2A = 10 + \log(a+b+c) + \log(a+b-c) - 2\log c$$

Given the excess of the hypothenuse over a side, and the difference of the angles; required the parts of the triangle.

Let $c - b = h, \quad A - B = M, \quad A + B = 90^\circ,$

$$A = \frac{90^\circ + M}{2}, \quad B = \frac{90 - M}{2}, \quad b = \frac{h . \cos A}{2 \sin^2 \frac{A}{2}} \text{ \&c.}$$

(80.) From the following table, in which all the parts of a right-angled triangle are given, any *data* may be chosen, and the preceding formulæ verified.

$$c = 128{\cdot}4327 \quad \log c = 2{\cdot}1086756$$
$$b = 66{\cdot}1364 \quad \log b = 1{\cdot}8204405$$
$$a = 110{\cdot}0951 \quad \log a = 2{\cdot}0417681$$
$$A = 59^\circ\, 0'\, 21''{\cdot}25 \qquad B = 30^\circ\, 59'\, 38''{\cdot}75$$
$$L . \sin A = L \cos B = 9{\cdot}9330925$$
$$L \cos A = L \sin B = 9{\cdot}7117649$$
$$L \tan A = L \cot B = 10{\cdot}2213276$$

(81.) We shall now proceed to the cases of oblique-angled triangles. The three angles are connected by any of the following relations.

$$A + B + C = 180^\circ \qquad A + B = 180^\circ - C \qquad \frac{A+B}{2} = 90^\circ - \frac{C}{2} \text{ \&c.}$$
$$\sin(A+B) = \sin C, \quad \cos(A+B) = -\cos C, \quad \tan(A+B) = -\tan C \text{ \&c.}$$
$$\sin \tfrac{1}{2}(A+B) = \cos \tfrac{1}{2} C, \quad \cos \tfrac{1}{2}(A+B) = \sin \tfrac{1}{2} C, \quad \tan \tfrac{1}{2}(A+B) = \cot \tfrac{1}{2} C \text{ \&c.}$$

Again, $\sin^2(A+B) = \sin^2 A . \cos^2 B + \cos^2 A . \sin^2 B + 2 \sin A \cos A \sin B \cos B$

for $\cos^2 A$ and $\cos^2 B$ write $1 - \sin^2 A$ and $1 - \sin^2 B$, which gives

$$\sin^2(A+B) = \sin^2 A + \sin^2 B + 2 \sin A . \sin B (\cos A \cos B - \sin A . \sin B)$$

But, $\sin(A+B) = \sin C \qquad \cos(A+B) = -\cos C$

or $\sin^2 C = \sin^2 A + \sin^2 B - 2 \sin A \sin B \cos C$

Similarly, $\sin^2 B = \sin^2 C + \sin^2 A - 2 \sin C \sin A \cos B$

$\sin^2 A = \sin^2 B + \sin^2 C - 2 \sin B \sin C \cos A$

Again, $\tan(A+B) = -\tan C = \dfrac{\tan A + \tan B}{1 - \tan A . \tan B}$

or $\tan A + \tan B + \tan C = \tan A . \tan B . \tan C$

(82.) Let $pU$, $qU$, and $rU$, be the three perpendiculars let fall from the vertices of the triangle upon the sides $aU$, $bU$, and $cU$. And let A′, B′, and C′ be the exterior angles of the triangle adjacent to A, B, and C. Then, if A be an obtuse angle, there are right-angled triangles, having for hypothenuses $bU$ and $cU$, and for sides opposite to A′, $r$ U and $q$ U. If A be a right angle we have $b = r$, and $c = q$. If A be an acute angle, we have right-angled triangles having for hypothenuses $b$ U and $c$ U, and for sides opposite to A, $r$ U and $q$ U. And the same of B and C. Again, $A + A' = 180°$, $B + B' = 180°$, $C + C' = 180°$, and we have

$$p = b \sin C \quad \text{or} \quad b \sin C' = b \sin C, \text{ in both cases}$$
$$p = c \sin B \quad \text{or} \quad c \sin B' = c \sin B, \text{ in both cases}$$

That is, $b \sin C = c \sin B$ or $\frac{b}{c} = \frac{\sin B}{\sin C}$

Similarly, $\frac{c}{a} = \frac{\sin C}{\sin A}$ and $\frac{a}{b} = \frac{\sin A}{\sin B}$

That is, any two sides are proportional to the sines of the opposite angles. This is the formula upon which all others relative to triangles will be made to depend.

(83.) Divide both sides of the value of $\sin^2 C$ in (81.) by $\sin^2 C$; substitute the ratios of the sides for those of the sines of angles, and we have

$$1 = \frac{a^2}{c^2} + \frac{b^2}{c^2} - 2\frac{a}{c} \cdot \frac{b}{c} \cdot \cos C$$

or $c^2 = a^2 + b^2 - 2ab \cos C$, $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

Similarly, $b^2 = c^2 + a^2 - 2ca \cos B$, $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$

$a^2 = b^2 + c^2 - 2bc \cos A$, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

These formulæ may be readily deduced from the triangle itself, it being obvious that $p = b \sin C$, and also that

$$c^2 = (b \sin C)^2 + (a - b \cos C)^2$$

(84.) We now proceed to put the preceding expressions in a form convenient for logarithmic computations:

$$\cos C = \frac{(a+b)^2 - c^2 - 2ab}{2ab} = \frac{(a+b)^2 - c^2}{2ab} - 1$$

Also $\cos C = \frac{(a-b)^2 - c^2 + 2ab}{2ab} = \frac{(a-b)^2 - c^2}{2ab} + 1$

or $1 + \cos C = \frac{(a+b+c)(a+b-c)}{2ab}$ $1 - \cos C = \frac{(c+a-b)(c+b-a)}{2ab}$

Let $a+b+c = 2s$ then $a+b-c = 2(s-c)$

$b+c-a = 2(s-a)$, $c+a-b = 2(s-b)$

Substitute these, and also the values of $1+\cos C$, &c. which gives

$$\cos^2\frac{C}{2} = \frac{s(s-c)}{ab} \qquad \sin^2\frac{C}{2} = \frac{(s-a)(s-b)}{ab}$$

Similarly
$$\cos^2\frac{B}{2} = \frac{s(s-b)}{ac} \qquad \sin^2\frac{B}{2} = \frac{(s-a)(s-c)}{ac}$$

$$\cos^2\frac{A}{2} = \frac{s(s-a)}{bc} \qquad \sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc}$$

$$\tan^2\frac{A}{2} = \frac{(s-b)(s-c)}{s(s-a)} \quad \tan^2\frac{B}{2} = \frac{(s-a)(s-c)}{s(s-b)} \quad \tan^2\frac{C}{2} = \frac{(s-a)(s-b)}{s(s-c)}$$

$$\sin A = \frac{2V}{bc} \qquad \sin B = \frac{2V}{ac} \qquad \sin C = \frac{2V}{ab}$$

where $V = \sqrt{\{s(s-a)(s-b)(s-c)\}}$

This is derived from the preceding, by aid of $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{2V}{abc}$$

$$\tfrac{1}{2}bc\sin A = \tfrac{1}{2}ca\sin B = \tfrac{1}{2}ab\sin C = V$$

(85.) To reduce $c^2 = a^2+b^2-2ab\cos C$ to a form adapted for logarithmic computation, proceed as follows:

1. $c^2 = (a+b)^2-2ab(1+\cos C)$

$$= (a+b)^2-4ab\cos^2\frac{C}{2} = (a+b)^2\left\{1-\frac{4ab}{(a+b)^2}\cos^2\frac{C}{2}\right\}$$

Now, $(a+b)^2-4ab$ being $(a-b)^2$ is positive; therefore $(a+b)^2$ is greater than $4ab$ and

$\frac{4ab}{(a+b)^2}$ and still more $\frac{4ab}{(a+b)^2}\cos^2\frac{C}{2}$ is less than 1.

Compute the positive square root of the last expression, and find in the tables the angle of which it is the sine; or find K from

$$\sin K = \frac{2\sqrt{ab}.\cos\frac{1}{2}C}{a+b}$$

Then $c^2 = (a+b)^2\{1-\sin^2 K\}$ or $c = (a+b)\cos K$

2. $c^2 = (a-b)^2+2ab(1-\cos C)$

$$= (a-b)^2\left\{1+\frac{4ab}{(a-b)^2}\sin^2\tfrac{1}{2}C\right\}$$

Compute K′ from $\tan \mathrm{K}' = \dfrac{2\sqrt{ab}\sin\frac{1}{2}\mathrm{C}}{a-b}$

Then $\quad c^2 = (a-b)^2\left\{1+\tan^2\mathrm{K}'\right\} \qquad c = \dfrac{a-b}{\cos \mathrm{K}'}$

(86.) Lastly, $\dfrac{a}{b} = \dfrac{\sin \mathrm{A}}{\sin \mathrm{B}}$ gives (54.) and (81.)

$$\frac{a-b}{a+b} = \frac{\sin \mathrm{A} - \sin \mathrm{B}}{\sin \mathrm{A} + \sin \mathrm{B}} = \frac{\tan\frac{1}{2}(\mathrm{A}-\mathrm{B})}{\tan\frac{1}{2}(\mathrm{A}+\mathrm{B})} = \frac{\tan\frac{1}{2}(\mathrm{A}-\mathrm{B}}{\cot\frac{1}{2}\mathrm{C}}$$

or
$$\tan\tfrac{1}{2}(\mathrm{A}-\mathrm{B}) = \frac{a-b}{a+b}\cot\tfrac{1}{2}\mathrm{C}$$

(87.) The preceding formulæ are sufficient for our general purpose. We shall now proceed to different cases.

The student may verify the methods as they are produced upon the sides, &c. of the following triangle:

$a = 15{\cdot}236 \quad \log a = 1{\cdot}1828710 \quad s-a = 3{\cdot}098 \quad \log(s-a) = 0{\cdot}4910814$

$b = 12{\cdot}414 \quad \log b = 1{\cdot}0939117 \quad s-b = 5{\cdot}920 \quad \log(s-b) = 0{\cdot}7723217$

$c = 9{\cdot}018 \quad \log c = 0{\cdot}9551102 \quad s-c = 9{\cdot}316 \quad \log(s-c) = 0{\cdot}9692295$

$s = 18{\cdot}334 \quad \log s = 1{\cdot}2632572 \quad \mathrm{V} = 55{\cdot}96866 \quad \log \mathrm{V} = 1{\cdot}7479449$

$a+b = 27{\cdot}650 \quad \log(a+b) = 1{\cdot}4416951 \quad a-b = 2{\cdot}822 \quad \log(a-b) = 0{\cdot}4505570$

$b+c = 21{\cdot}432 \quad \log(b+c) = 1{\cdot}3310627 \quad b-c = 3{\cdot}396 \quad \log(b-c) = 0{\cdot}5309677$

$a+c = 24{\cdot}254 \quad \log(a+c) = 1{\cdot}3847834 \quad a-c = 6{\cdot}218 \quad \log(a-c) = 0{\cdot}7936507$

| | | L. sin. | L. cos. | L. tan. |
|---|---|---|---|---|
| A | = 89° 9′ 23″·54 | 9·9999530 | 8·1679268 | 11·8320262 |
| B | = 54 33 25·12 | 9·9109937 | 9·7633479 | 10·1476458 |
| C | = 36 17 11·48 | 9·7721922 | 9·9063714 | 9·8658208 |
| ½A | = 44 34 41·77 | 9·8462647 | 9·8526583 | 9·9936064 |
| ½B | = 27 16 42·56 | 9·6611649 | 9·9487989 | 9·7123661 |
| ½C | = 18 8 35·74 | 9·4933102 | 9·9778520 | 9·5154583 |
| ½(A—B) | = 17 17 59·21 | 9·4722989 | 9·9798951 | 9·4924039 |
| ½(B—C) | = 9 8 6·82 | 9·2007551 | 9·9944564 | 9·2062988 |
| ½(A—C) | = 26 26 6·03 | 9·6485379 | 9·9520365 | 9·6965016 |
| $\mathrm{K}_a$ | = 44 41 30·6 | 9·8471366 | 9·8518083 | .......... |
| $\mathrm{K}_b$ | = 59 12 50·4 | 9·9340361 | 9·7091283 | .......... |
| $\mathrm{K}_c$ | = 70 57 53·6 | 9·9755783 | 9·5134140 | .......... |
| $\mathrm{K}'_a$ | = 77 7 15·5 | ......... | 9·3408970 | 10·6408380 |
| $\mathrm{K}'_b$ | = 59 56 28·9 | ......... | 9·6997390 | 10·2375348 |
| $\mathrm{K}'_c$ | = 71 45 50·9 | ......... | 9·4954464 | 10·4821746 |

where, by $\mathrm{K}_a$ is meant the angle K of (85.), as obtained when $a$ is to be found, or from $a^2 = b^2 + c^2 - 2bc\cos A$. In the following table, $p$U, $q$U, and $r$U, are, as before, the perpendiculars on $a$, $b$, and $c$, and $a_b$U means the segment of $a$U adjacent to $b$U, &c.

$p = 7{\cdot}347 \quad \log p = 0{\cdot}8661039$

$q = 9{\cdot}017 \quad \log q = 0{\cdot}9550632$

$r = 12{\cdot}413 \quad \log r = 1{\cdot}0938647$

$a_b = 10{\cdot}006 \quad \log a_b = 1{\cdot}0002831 \qquad a_c = 5{\cdot}230 \quad \log a_c = 0{\cdot}7184581$

$b_a = 12{\cdot}281 \quad \log b_a = 1{\cdot}0892424 \qquad b_c = {\cdot}133 \quad \log b_c = 9{\cdot}1230370$

$c_a = 8{\cdot}835 \quad \log c_a = 0{\cdot}9462189 \qquad c_b = {\cdot}183 \quad \log c_b = 9{\cdot}2618385$

In all the following article, $a$ is the greatest side, $b$ the mean, and $c$ the least. Consequently, A is the greatest angle, &c. { } means that the quantity enclosed is the one found by the process; all the others in it being given or previously found.

(88.) First Case. Given the three sides, required the angles.

*First method.* (6.) $a_b^2 - a_c^2 = b^2 - c^2$ or $(a_b - a_c)a = (b+c)(b-c)$

$$\log\{a_b - a_c\} = \log(b+c) + \log(b-c) - \log a$$

Hence, $a_b - a_c$ is found: let it be $h$.

$$\{a_b\} = \tfrac{1}{2}(a+h) \qquad \{a_c\} = \tfrac{1}{2}(a-h)$$

$$\log.\cos\{C\} = \log a_b - \log b \qquad \log.\cos(B) = \log a_c - \log c$$

$$\{A\} = 180^\circ - (B+C)$$

This method is the shortest when *all* the angles are wanted, but should not be used (68.) when one of the angles is very small; or one of the sides very small in proportion to the rest. When one angle only is wanted, use the

*Second method.* To find A, use (84.) one of these,

$$\text{L.sin}\{\tfrac{1}{2}A\} = 10 + \tfrac{1}{2}(\log\overline{s-b} + \log\overline{s-c} - \log b - \log c)$$

$$\text{L.cos}\{\tfrac{1}{2}A\} = 10 + \tfrac{1}{2}(\log s + \log\overline{s-a} - \log b - \log c)$$

$$\text{L.tan}\{\tfrac{1}{2}A\} = 10 + \tfrac{1}{2}(\log\overline{s-b} + \log\overline{s-c} - \log s - \log(s-a))$$

$$\{A\} = 2 \times \tfrac{1}{2}A$$

(89.) Second Case. Given two sides ($a$ and $b$, $a$ the greater), and the included angle C; required the rest.

*First method.* When both the other angles and the remaining side are required, use

$$\mathrm{L.tan}\left\{\tfrac{1}{2}\overline{\mathrm{A}-\mathrm{B}}\right\} = \log(a-b) + \mathrm{L.cot}\,\tfrac{1}{2}\mathrm{C} - \log(a+b)^{*}$$

$$\left\{\tfrac{1}{2}\overline{\mathrm{A}+\mathrm{B}}\right\} = 90^\circ - \tfrac{1}{2}\mathrm{C}$$

$$\{\mathrm{A}\} = \tfrac{1}{2}(\mathrm{A}+\mathrm{B}) + \tfrac{1}{2}(\mathrm{A}-\mathrm{B}) \qquad \{\mathrm{B}\} = \tfrac{1}{2}(\mathrm{A}+\mathrm{B}) - \tfrac{1}{2}(\mathrm{A}-\mathrm{B})$$

$$\log\{c\} = \mathrm{L.sin\,C} + \log a - \mathrm{L\,sin\,A} \quad \dagger$$

*Second method.* When the side only is required (85.), use either of the following:

$$\left.\begin{aligned}\mathrm{L.sin}\{\mathrm{K}\} &= \tfrac{1}{2}(\log a + \log b) + \log 2 + \mathrm{L\,cos}\,\tfrac{1}{2}\mathrm{C} - \log(a+b)\\ \log\{c\} &= \log(a+b) + \mathrm{L.cos\,K} - 10\end{aligned}\right\}$$

$$\left.\begin{aligned}\mathrm{L.tan}\{\mathrm{K}'\} &= \tfrac{1}{2}(\log a + \log b) + \log 2 + \mathrm{L.sin}\,\tfrac{1}{2}\mathrm{C} - \log(a-b)\\ \log\{c\} &= \log(a-b) + 10 - \mathrm{L.cos\,K}'\end{aligned}\right\}$$

*Third method.* When the given angle is very nearly 180°, let it be $180^\circ - \mathrm{C}_,$, where $\mathrm{C}_,$ is small; we have then

$$c^2 = a^2 + b^2 - 2ab\cos(180^\circ - \mathrm{C}_,) = a^2 + b^2 + 2ab\cos\mathrm{C}_,$$

$$= (a+b)^2 - 4ab\sin^2\tfrac{1}{2}\mathrm{C}_, = (a+b)^2\left\{1 - \frac{4ab}{(a+b)^2}\sin^2\tfrac{1}{2}\mathrm{C}_,\right\}$$

By the binomial theorem $\sqrt{1-x} = 1 - \frac{1}{2}x$ nearly, $x$ being small

$$c = (a+b)\left(1 - \frac{2ab}{(a+b)^2}\cdot\sin^2\tfrac{1}{2}\mathrm{C}_,\right) = a + b - \frac{2ab}{a+b}\sin^2\tfrac{1}{2}\mathrm{C}_,$$

very nearly. But $\sin\mathrm{C}_, = 2\sin\frac{1}{2}\mathrm{C}_,.\cos\frac{1}{2}\mathrm{C}_,$, or $\cos\frac{1}{2}\mathrm{C}_,$ being very nearly 1, we have

$$\sin\tfrac{1}{2}\mathrm{C}_, = \tfrac{1}{2}\sin\mathrm{C}_,, \quad \sin^2\tfrac{1}{2}\mathrm{C}_, = \tfrac{1}{4}\sin^2\mathrm{C}_,, \text{ very nearly:}$$

$$c = a + b - \tfrac{1}{2}\frac{ab\sin^2\mathrm{C}_,}{a+b} \quad \text{very nearly}$$

$$\log\{k\} = 2\,\mathrm{L\,sin\,C}_, + \log a + \log b - \log(a+b) - 20$$

$$c = a + b - \tfrac{1}{2}k$$

*Fourth method.* When $b$ is very small compared with $a$, and the small angle B is wanted, we have

$$\frac{a}{b} = \frac{\sin\mathrm{A}}{\sin\mathrm{B}} = \frac{\sin(\mathrm{B}+\mathrm{C})}{\sin\mathrm{B}} = \cos\mathrm{C} + \cot\mathrm{B}.\sin\mathrm{C}$$

---

* 10 is not added here to give the tabular logarithm, because L. cot is already too great by 10.

† The excess of the one tabular log. compensates that of the other.

$$\cot B = \frac{a - b\cos C}{b\sin C} \qquad \tan B = \frac{b\sin C}{a - b\cos C} = \frac{b}{a}\sin C$$

very nearly; hence $b$ is readily found, very nearly.

(90.) THIRD CASE. Given two sides and an angle *not included*, required the rest.

It can be shewn immediately that this is a problem of the second degree, admitting sometimes of two solutions. Let $a$ and $b$ be the given sides, B the given angle; we have then

$$b^2 = a^2 + c^2 - 2ac\cos B$$

or

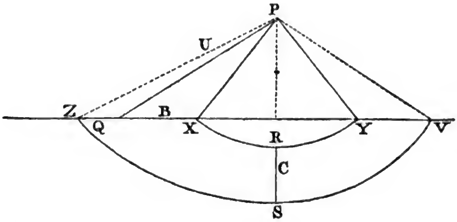
$$\left\{c\right\} = a\cos B \pm \sqrt{b^2 - a^2\sin^2 B}$$

$$\sin\left\{C\right\} = \frac{c\sin B}{b} \qquad \sin\left\{A\right\} = \frac{a\sin B}{b}$$

1. There is no such triangle at all when $b$ is less than $a\sin B$.

2. Only positive values of $c$ must be taken, for a negative value of C would give $\sin C$ negative, or C greater than two right angles, which is impossible in a triangle. The roots of the equation are both positive (*Algebra*, p. 139.) when $a^2 - b^2$ is positive, or $a$ greater than $b$; in this case there are two triangles satisfying the conditions in question. When $a = b$ one triangle disappears, for then one value of $c$ is 0; when $a$ is less than $b$ there is only one solution.

The geometrical construction will also shew this. Lay down the angle B equal to the given angle, and take $PQ = aU$. With



centre P and radius $PY = bU$ describe a circle, then, if $bU$ be less than PR or $a\sin B.U$, there is no such triangle; if $bU = a\sin B.U$, there is a right-angled triangle QPR, under the given conditions; if, PY being greater than PR, it be still less than PQ, there are two triangles, QPX, and QPY, satisfying the conditions. But if PY be greater than PQ there is only one such triangle, QPV, for the triangle QPZ obviously has not the given angle, but its supplement.

For logarithmic solution, proceed as follows :

$$\mathrm{L.\,sin}\{A\} = \log a + \mathrm{L.\,sin}\,B - \log b$$

Here (50.) are two angles which satisfy the conditions; one less than a right angle, which call $A_{,}$, the other $A_{,,}$, or $180° - A_{,}$, greater. Or it may happen that $a\dfrac{\sin B}{b}$ may be greater than unity, in which case L. sin A will be found greater than 10, or log. sin A positive, and there is no such angle. In the case where this does not happen, we proceed as follows :

Either $\{C\} = 180° - A_{,} - B$ or $180 - A_{,,} - B$

Or $C_{,} = 180 - A_{,} - B$ $C_{,,} = A_{,} - B$

if $A_{,}$ be greater than B; for otherwise, $C_{,,}$ is negative, and is here inadmissible. Consequently, the two values of $c$ being $c_{,}$ and $c_{,,}$, we have,

$$c_{,} = \frac{b\sin C_{,}}{\sin B} = \frac{b\sin(A_{,}+B)}{\sin B} \qquad c_{,,} = \frac{b\sin(A_{,}-B)}{\sin B}$$

$$\log\{c_{,}\} = \log b + \mathrm{L}\sin C_{,} - \mathrm{L}\sin B$$

$$\log\{c_{,,}\} = \log b + \mathrm{L}\sin C_{,,} - \mathrm{L}\sin B \text{ (if } C_{,,} \text{ be positive)}$$

(91.) Fourth Case. Given a side and two angles; ($a$ U the side), required the rest.

$$\{\text{Third angle}\} = 180° - (\text{sum of given angles})$$

$$\log b = \log a + \mathrm{L}\sin B - \mathrm{L}\sin A$$

$$\log c = \log a + \mathrm{L}\sin C - \mathrm{L}\sin A$$

# CHAPTER IV.

## ON THE EXTENSION OF THE MEANINGS OF SYMBOLS, AND ON THE SQUARE ROOTS OF NEGATIVE QUANTITIES.

(92.) We have as yet explicitly used only two different kinds of symbols: those of *quantity*, specific and general (arithmetical and algebraical), and those of *operation;* specific, as + —, &c. and general, $f$, $\phi$, &c. (*Algebra*, p. 203). But we are not therefore bound never to use any other symbols; the only laws by which our right to such aids is limited, are the following.

1. Neither the symbols themselves, nor any expressions in which they are used, must have different meanings of any such kind, such that the consequences of one meaning may be confounded with, and used for, the consequences of another.

2. The consequences of all assumptions must follow logically from the assumptions themselves.

It therefore becomes of interest to consider in what other possible ways we might use symbols. And first it must strike us that all yet employed may be styled under one name, more general than either operation or quantity. They are, in fact, symbols of *discrimination* or *distinction*. Thus, in $a$, $b$, and $c$, in which the symbolic difference is only difference of shape, that circumstance is made the distinction between difference of numerical magnitude. In + and — the same distinction, namely, of form, is made that of the direction given, as to which of two fundamental operations is to be performed. In $ab$ and $\frac{a}{b}$ we see that difference of position, with the employment of a new, and not altogether necessary, symbol, is the distinction which implies difference of operations.

(93.) Let us now look at the extension of arithmetic into algebra, not confining ourselves to the notion of *operation* or *quantity*, but generalising our idea of symbols to that of mere discrimination; the object being to consider, whether, in this point of view, extension is *possible*, preliminary to the further question of whether it is *advisable*.

Looking narrowly at the steps by which we ascended (*Algebra*, p. 57. and 110.), we see that, so far as the discriminative quality of the symbols is concerned, we found the following consequence of our fundamental and *arithmetical* definitions of $+$ $-$ &c. namely, that *old rules were sufficient to express new distinctions* consistently with the old distinctions; in such sort that, whenever the new distinctions disappeared, the result was a legitimate consequence of the older notions, such as would have been obtained if the more circuitous method had been adopted, to avoid which the new distinctions were introduced. For example, when we came to the expressions $a+(+b)$ and $a+(-b)$, which have no meaning under the original meaning of $+$ and $-$, we had extended our ideas to the following question. The distinction between a magnitude of any one kind and its diametrically opposite, being denoted by $^{*}b$ and $\P b$, if there occur a case of a problem in which $^{*}b$ requires to be added to $a$, what will the similar problem require, in which $\P b$ is used where $^{*}b$ was used in the other. And we found, 1. That where $^{*}b$ requires addition, $\P b$ requires subtraction. 2. That $+a$ and $-a$ would themselves consistently express $^{*}a$ and $\P a$, provided that the rules appertaining to the original meaning of $+$ and $-$, *and no others*, should be also applied to them in their new and distinctive capacity. And we also found that the disappearance of $+$ and $-$, in their new character, was accompanied by the disappearance of all traces of the new distinctions; or that all theorems which in any case took the old forms only, were true under the old meanings. The old algebra (general arithmetic) was, in every sense, part of the new one.

(94.) Now, this we shall lay down as the restriction under which any *further* extension is to be made; namely, that all theorems at present existing, are to be theorems which are true, whenever they are the consequences of any further extension; and true in the sense in which they exist at present. We might propose infinite numbers of changes of meaning, under which *some* theorems would remain true, but in which others would not be so. For instance, if $\times$ placed between two quantities, were made to signify that their *sines* (not the numbers themselves) should be multiplied together, then (61.)

$$a \times a - b \times b = (a+b) \times (a-b)$$

would be true; but $a \times 2b = b \times 2a$ would not be true.

(95.) There might arise cases in which the answers of problems could not be expressed without more power of symbols than is possessed at present. For instance, will any one undertake to say, that of all possible problems, there is no one of which the answer is as follows: Divide 6754321 by 12, in the common way, with this exception, that whenever the remainder is an even number, it is to stand, but whenever it is an odd number, it is to be increased by 1, if the preceding remainder were even, and diminished by 1 if odd. The answer to this question would be,

$$12\overline{)6754321}$$
$$570367\tfrac{8}{12}$$

but no symbols, at present possessed, would describe the operation. Again, if we ask what is that expression which, when $x$ is positive, is $x^2$, but when $x$ is negative, is $x^3$? This cannot be expressed by present symbols; and the same may be said of many imaginable results. So much for the possibility of further extension, or a new symbol of distinction; the next question must be, is it wanted, and can it be made?

(96.) When the earlier algebraists first began to occupy themselves with questions expressed in general terms, the difficulties of subtraction soon became obvious, inasmuch as the greater would sometimes demand to be subtracted from the less. The science has been brought to its present state through three distinct steps. The first was tacitly to contend for the principle that human faculties, at the outset of any science, are judges both of the extent to which its results can be carried, and of the form in which they are to be expressed. *Ignorance*, the necessary predecessor of knowledge, was called *nature*; and all conceptions which were declared unintelligible by the former, were supposed to have been made impossible by the latter. The first who used algebraical symbols in a general sense, Vieta, concluded that subtraction was a defect, and that expressions containing it should be in every possible manner avoided. *Vitium negationis*, was his phrase. Nothing could make a more easy pillow for the mind, than the rejection of all which could give any trouble; but if Euclid had altogether dispensed with the *vitium parallelorum*, his geometry would have been confined to twenty-six propositions of the first book.

The next and second step, though not without considerable fault, yet avoided the error of supposing that the learner was a competent critic. It consisted in treating the results of algebra as necessarily true, and as representing some relation or other, however inconsistent they might be with the suppositions from which they were deduced. So soon as it was shewn that a particular result had no existence as a quantity, it was permitted, by definition, to have an existence of another kind, into which no particular inquiry was made, because the rules under which it was found that the new symbols would give true results, did not differ from those previously applied to the old ones. A symbol, the result of operations upon symbols, either meant quantity, or nothing at all; but in the latter case it was conceived to be a certain new kind of quantity, and admitted as a subject of operations, though not one of distinct conception. Thus, $1-2$, and $a-(a+b)$, appeared under the name of negative quantities, or quantities less than nothing. These phrases, incongruous as they always were, maintained their ground, because they always produced true results, whenever they produced any result at all which was intelligible: that is, the quantity less than nothing, in defiance of the common notion that all conceivable quantities are greater than nothing, and the square root of the negative quantity, an absurdity constructed upon an absurdity, always led to truths when they led back to arithmetic at all, or when the inconsistent suppositions destroyed each other. This ought to have been the most startling part of the whole process. That contradictions might occur, was no wonder; but that contradictions should uniformly, and without exception, lead to truth in algebra, and in no other species of mental occupation whatsoever, was a circumstance worthy the name of a mystery.

Nothing could prevail against the practical result, that theorems so produced were true; and at last, when the interpretation of the abstract negative quantity shewed that a part, at least, of the difficulty admitted of rational solution, the remaining part, namely, that of the square root of a negative quantity, was received, and its results admitted, with increased confidence.

(97.) The complete explanation of the embarrassing circumstances is comparatively modern; the latter arise from a very simple logical misconception, the assumption of the truth of a converse, namely, that if B follow from A, B follows from nothing else but A; or if A always yield B, B when it appears, must have been produced from A. We can imagine $a, b, c$, &c. $+$, $-$, $\times$, &c., defined in many different ways, so that certain of the theorems of algebra should severally be true under more than one set of meanings; and we have shewn an instance in page 72. We can further imagine it possible that one set of meanings should be so connected with another, that all theorems which are true in the first, should also be true in the second, and that, beside this, there should be other classes of theorems which are true in the second, but not true in the first. This is as possible as that one figure should be entirely contained in another, without filling it. We might thus conceive a succession of extensions of definitions, giving a series of sciences, each containing the whole of its predecessors, and more. Whence it is clear, that if the methods of operation, under any science, wandered beyond its limits, without that corresponding extension of definitions being made which converted the logic of one science into that of the other, the consequence would be, the appearance of some of the symbolic means of expressing truths in the wider science, without the key to their interpretation. This happened when we first came to the negative symbol in algebra (*Algebra*, p. 12), and we were under the necessity of adopting more extensive definitions. The proof of undue extension in the operations again occurred (p. 110.), where $\sqrt{-1}$ first appeared; but it was not then necessary to follow up the extensions necessary for its elucidation.

This matter is one of difficulty to a beginner, unused to the idea of a finished language having the meanings of all its terms extended so that the old meanings are only part of the new ones. But, in reality, he has gone through the process, by insensible steps, in his childhood.* Let him compare the first impression he was made to receive by the words "I see," with the sense he puts upon the phrase when, if he understand the preceding, he says he *sees* my meaning. That which I here call the extended use of the term, he will call the allegorical or metaphorical mode, that is, if we translate these Greek terms, the *other-speaking*, or *transferred* mode of expression. But in what way is the speech changed? By using the word to "see," not only as denoting perception by the eyes, but perception by the understanding in any way whatever. If we *heard* any one speak, we might still *see* his meaning.

(98.) Let us consider the most general meaning of any fundamental equation of algebra: for instance, $a + b = b + a$. We restrict ourselves to this only, that $b$ and $a$, and $+$, mean the same thing in both places, that $=$ denotes, *gives the same result as*, or *is the same in effect as*, and that the order of the expression is from left to right, that is, on the first side we first examine $a$, and $b$ on the second side. Then $a+b$ merely implying that something, $b$, is done in some manner, $+$, after something else, $a$, has been done; the above shews no more than that it is indifferent whether $a$ is done first or $b$, or that the result is the same in both cases. Consistently with $a+b=b+a$, many meanings might be shewn to be impossible, and many possible. Neither need $a$ and $b$ stand in any way for quantities: for instance, the preceding would be true if $a$ stood for a line not considered as a length, $b$ for another line, $+$ for the formation of a rectangle out of $a$ and $b$, and $=$ for equality of space enclosed. The preceding relation may therefore be a truth under an infinite number of different meanings. Let us now take another form, $ab=ba$, which admits of exactly as many meanings as the first, and denotes merely indifference of order. Suppose we pick out two such meanings at pleasure, and assign them, only requiring that $a$ and $b$ shall mean the same in both relations. We then have definitions for $a$, $b$, $+$ and the meaning of juxtaposition, and have explained $a+b=b+a$, and $ab=ba$, from which it follows that if $+$, as defined, will intelligibly apply to $ab$, we have $ab+b=b+ab$, &c. But if we would have our new algebra identical with the old one in forms, we must choose such meanings for the symbols in the two relations as will also make

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* The similarity of the views here given, with some in the review of Mr. Peacock's "Algebra," in the ninth volume of the "Journal of Education," makes it necessary for the author to state that he was also the writer of those articles.

$$a(a+b)=aa+ab \text{ represent a new truth.}$$

This is a restriction upon all the possible allowances of meaning which might be made: for it does not follow that every meaning which makes $a+b=b+a$, and $ab=ba$, true, also makes the last true. And other relations might be introduced which would still more restrict the meanings, and so on, until every fundamental relation necessary to algebra had been considered. If we could really collect all the possible meanings of each separate relation, and find the method of ascertaining which must be struck out for each and every new combination which the mechanism of algebra introduces, we should, if we could classify the remainder uniting those which are particular cases of a general meaning under their general head, be left with an *algebra* in the widest possible sense of the word.

*We do not want to change operations; but we want to find all the definitions under which those operations will demonstratively enable us to pass from one truth to another.*

(99.) Two explanations may be given of the manner in which $\sqrt{-1}$ may rationally be used; the first purely symbolical, that is, employing $\sqrt{-1}$ as a symbol, the meaning of which is given for convenience only; the second derived from geometry, and an extension of the method by which lines measured in opposite directions are represented by letters with different signs. The first is wholly algebraical; the second (for a reason which will afterwards appear) is an application of geometry to algebra.

Let $k$ be a symbol which does not stand for quantity, but for a distinction, in whatever way it may be required to distinguish; that is, $ka$ and $a$ both stand for the same magnitude, but the first has the mark of either being used in a certain way, or appropriated for certain purposes, or liable to be rejected under certain circumstances, or whatever other distinction $k$ may indicate. When two separated terms are multiplied together, as $ka$ and $kb$, let the product be written $k^2ab$, in which $k^2$ merely implies the presence, in a product, of two terms which had the mark of distinction. Similarly, $k^3a\ c$ is the distinction of a product made up of three terms which had the distinction, and so on. Let $k(a+b)$ be the distinction between $a+b$ and $ka+kb$, and so on. We are at liberty to assign to $k$ any *discriminative power* we please. Let it be as follows: $k$ is to be a distinction which ceases altogether in terms marked with $k^4$, $k^8$, $k^{12}$, &c. or $k^{4n}$, and with $k^{-4}$, $k^{-8}$, $k^{-12}$, &c. or $k^{-4n}$, and which is preserved in $k^5$, $k^9$, $k^{13}$, &c. $k^{-3}$, $k^{-7}$, $k^{-11}$, &c. or in $k^{4n+1}$ where $n$ is any whole number positive or negative. And let $k^2$, $k^6$, $k^{10}$, &c. $k^{-2}$, $k^{-6}$, $k^{-10}$, &c. be distinctions amounting to a change of sign in the terms denoted by them; so that if for $k^2a$ we write $-a$, the object of the distinction is fulfilled, and the term need no longer be distinguished. Let $k^3$, $k^7$, $k^{11}$, &c. or $k^{-1}$, $k^{-5}$, $k^{-9}$, &c. imply both a change of sign and also the continuance of the distinction denoted by $k$; so that $k^3a$ means $-ka$. We have then defined every thing except $k$ itself, by the following identities;

$$k^{4n}a \text{ means } a, \quad k^{4n+1}a \text{ means } ka,$$
$$k^{4n+2}a \text{ means } -a, \quad k^{4n+3}a \text{ means } -ka$$

And thus every algebraical expression, when its distinctions are all marked, is reducible to the form $P+kQ$ where P and Q are algebraical. Thus,

$$a_0+a_1k+a_2k^2+\ \ldots \text{ means } a_0-a_2+a_4+\ \cdots\ +k(a_1-a_3+a_5-\&c.)$$

Now $k$ itself is to have meaning as follows: in the equation

$$P+kQ = P'+kQ'$$

let the presence of $k$ indicate that equality will still remain as well between the parts independent of $k$, as between those affected by $k$. If we had merely $P+Q=P'+Q'$ it would not at all follow that $P=Q$ and $P'=Q'$: but when we mean to make this additional supposition, let us signify the same by the presence of $k$.

Now, it will be obvious upon looking from (70.) to (73.) that we have here only made a notation to *express* distinctions which have been actually *arrived* at by process of reasoning. We found a method of embodying all the results previously obtained, of this kind: $2\cos\theta$ cannot be $x+\frac{1}{x}$; but if we work in any manner with $x+\frac{1}{x}$ and $x-\frac{1}{x}$, and produce an equation, then that same equation will correspond to one or two true equations, if we work in precisely the same manner with $2\cos\theta$ and $2k\sin\theta$, and then let $k$ have its discriminative powers. And we shall then find that the result is the same as if

$2\cos n\theta$ had taken the place of $x^n+\frac{1}{x^n}$ and $2k\sin n\theta$ of $x^n-\frac{1}{x^n}$

When none but even numbers of $k$s occur, it is obvious that the result can be only one equation of the form $P=P'$; but when odd numbers of $k$s also occur, the result will be of the form $P+kQ=P'+kQ'$, giving two equations, $P=P'$, and $Q=Q'$.

(100.) Next, observe that the algebraical symbol $\sqrt{-1}$, which is certainly no quantity, positive or negative, and therefore not to be reasoned upon as a quantity, yet has this property, that if those rules be applied which would have been applied had it been a quantity, the results will be expressive of the distinction denoted by $k$. For in that case we have

$\sqrt{-1}$ is $\sqrt{-1}$; $(\sqrt{-1})^2$ is $-1$; $(\sqrt{-1})^3$ is $-\sqrt{-1}$;

$(\sqrt{-1})^4$ is 1; $(\sqrt{-1})^5$ is $\sqrt{-1}$; $(\sqrt{-1})^6$ is $-1$;
$(\sqrt{-1})^7$ is $-\sqrt{-1}$; $(\sqrt{-1})^8$ is 1.

Again,
$(\sqrt{-1})^{-1}$ is $-\sqrt{-1}$; $(\sqrt{-1})^{-2}$ is $-1$; $(\sqrt{-1})^{-3}$ is $\sqrt{-1}$, &c. from which the coincidence is apparent. Therefore, by making $\sqrt{-1}$ the symbol of the distinction meant by $k$, all the remaining distinctions will be drawn by applying the common rules of algebra to $\sqrt{-1}$ as if it were a quantity.

When in (71.) we began to compare $\cos\theta + k\sin\theta$ with $\varepsilon^{k\theta}$, we were also, in fact, comparing it with $1 + k\theta + k^2\frac{\theta^2}{2} + \ldots.$, or (if we apply the meaning of $k$) with

$$1 - \frac{\theta^2}{2} + \frac{\theta^4}{2.3.4.} + \ldots. + k\left(\theta - \frac{\theta^3}{2.\ 3} + \frac{\theta^5}{2.3\ 4.5} + \ldots.\right)$$

Now, if the method of proceeding be valid, which extends the distinctive property of $k$ to the developement of $\varepsilon^{k\theta}$, then

$$\cos\theta + k\sin\theta = 1 - \frac{\theta^2}{2} + \ldots. + k\left(\theta - \frac{\theta^3}{2.3} + \ldots.\right)$$

or $$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{2.3.4} - \frac{\theta^6}{2.3.4.5.6} + \ldots.$$

$$\sin\theta = \theta - \frac{\theta^3}{2.3} + \frac{\theta^5}{2.3.4.5} - \frac{\theta^7}{2.3.4.5.6.7} + \ldots.$$

(101.) We shall now go through a strict deduction of these equations, which will shew that what we have done would, had we seen how, itself have been one. It is evident that

$$\cos\theta + k\sin\theta = \cos\theta + k\sin\theta \left\{\begin{matrix}\text{which amounts to}\\ \cos\theta = \cos\theta,\ \sin\theta = \sin\theta\end{matrix}\right.$$

Square both sides, which gives

$$(\cos\theta + k\sin\theta)^2 = \cos^2\theta + k\,.\,2\sin\theta\,.\cos\theta + k^2\sin^2\theta$$
$$= \cos^2\theta - \sin^2\theta + k\,.\,2\sin\theta\,.\cos\theta \quad = \quad \cos 2\theta + k\sin 2\theta$$

Generally, if $\cos n\theta + k\,.\sin n\theta = (\cos\theta + k\sin\theta)^n$

$$\times\,(\cos\theta + k\sin\theta) \text{ and } \left.\begin{matrix}\cos n\theta\,.\cos\theta\\ +k^2\sin n\theta\,.\sin\theta\end{matrix}\right\} + k\left\{\begin{matrix}\cos n\theta\,.\sin\theta\\ +\sin n\theta\,.\cos\theta\end{matrix}\right. = (\cos\theta + k\sin\theta)^{n+1}$$

or $$\cos(n+1)\theta + k\,.\sin(n+1)\theta = (\cos\theta + k\sin\theta)^{n+1}$$

So that this relation, if true for one whole value of $n$, is true for the next. But it is true for $n = 1$ and $n = 2$, therefore it is true for all. This is called *De Moivre's Theorem*.

Develope the second side, substitute the meanings of $k^2$, $k^3$, &c., and form the two resulting equations, which will be found to be

$$\cos n\theta = \mathrm{c}^n - n\frac{n-1}{2}\mathrm{c}^{n-2}\mathrm{s}^2 + n\,.\frac{n-1}{2}\,\frac{n-2}{3}\,\frac{n-3}{4}\mathrm{c}^{n-4}\mathrm{s}^4 - \;\ldots.$$

$$\sin n\theta = n\mathrm{c}^{n-1}\mathrm{s} - n\frac{n-1}{2}\,\frac{n-2}{3}\mathrm{c}^{n-3}\mathrm{s}^3 + n\frac{n-1}{2}\,\frac{n-2}{3}\,\frac{n-3}{4}\,.\,\frac{n-4}{5}\mathrm{c}^{n-5}\mathrm{s}^5 - \;..$$

where c means $\cos\theta$, and s, $\sin\theta$. Divide both sides by $\mathrm{c}^n$, and put t ($\tan\theta$) for $\frac{\mathrm{c}}{\mathrm{s}}$; at the same time divide and multiply $n$, $n\frac{n-1}{2}$, &c. by a power of $n$ of the same number of factors, which gives

$$\frac{\cos n\theta}{\mathrm{c}^n} = 1 - \frac{1-\frac{1}{n}}{2}(n\mathrm{t})^2 + \frac{1-\frac{1}{n}}{2}\,\frac{1-\frac{2}{n}}{3}\,\frac{1-\frac{3}{n}}{4}\,(n\mathrm{t})^4 - \;\ldots.$$

$$\frac{\sin n\theta}{\mathrm{c}^n} = n\mathrm{t} - \frac{1-\frac{1}{n}}{2}\,\frac{1-\frac{2}{n}}{3}(n\mathrm{t})^3 + \frac{1-\frac{1}{n}}{2}\,\frac{1-\frac{2}{n}}{3}\,\frac{1-\frac{3}{n}}{4}\,\frac{1-\frac{4}{n}}{5}\,(n\mathrm{t})^5 - \;\ldots.$$

which is true for all whole values of $n$, and all values of $\theta$. Now, consider all those whole values of $n$, and values of $\theta$, which make $n\theta = x$, a given angle: whence (*Algebra*, p. 157.) the limits of the two sides of each of the preceding, made by increasing $n$ without limit, will be equal. We proceed to find these limits. We have

$\mathrm{c}^n = \left(\cos\frac{x}{n}\right)^n$ the limit of which (70.) is 1; $\frac{1}{n}$, $\frac{2}{n}$, $\frac{3}{n}$, &c. the limits of which are severally 0; $n\mathrm{t}$ or $n\tan\frac{x}{n}$ or $x\left(\tan\frac{x}{n} \div \frac{x}{n}\right)$ the limit of which (48.) is $x \times 1$, or $x$; and $n\theta$, which is $x$ throughout.

Take the limits of both sides, which give

$$\left.\begin{aligned}\cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \text{\&c.}\\ \sin x &= x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \text{\&c.}\end{aligned}\right\}\ \text{the theorem in question.}$$

Hence we have

$$\cos\theta + k\sin\theta = 1 - \frac{\theta^2}{2} + \cdot\cdot + k\left(\theta - \frac{\theta^3}{2.3} + \ldots.\right)$$

$$= 1 - k\theta + k^2\frac{\theta^2}{2} + k^3\frac{\theta^3}{2.3} + \ldots. = \varepsilon^{k\theta}$$

if we abbreviate the preceding series into the formula of which it would be the algebraical developement, if $k$ were a quantity. We shall now adopt the symbol $\sqrt{-1}$ for $k$, which gives (for all values of $\theta$)

$$\cos\theta + \sin\theta.\sqrt{-1} = \varepsilon^{\theta\sqrt{-1}}$$

For $\theta$ write $-\theta$ (44.) $\cos\theta - \sin\theta.\sqrt{-1} = \varepsilon^{-\theta\sqrt{-1}}$

$$\cos\theta = \frac{\varepsilon^{\theta\sqrt{-1}} + \varepsilon^{-\theta\sqrt{-1}}}{2} \qquad \sin\theta = \frac{\varepsilon^{\theta\sqrt{-1}} - \varepsilon^{-\theta\sqrt{-1}}}{2}$$

that is, if we develope the preceding exponential expressions, paying attention to the difference of developement denoted by $\sqrt{-1}$, $(\sqrt{-1})^2$ &c. we shall arrive at series which we have shewn to be the developements of $\sin\theta$ and $\cos\theta$.

(102.) Some marks of distinction are definition, and all the rest are consequences. Thus $\varepsilon^{k\theta}$ has $k$ a symbol of distinction; but it does not mean $k.\varepsilon^{\theta}$, but $\cos\theta + k\sin\theta$. And there are some results which when enunciated as if $\sqrt{-1}$ were a quantity, are yet more inconceivable than less than nothing, in an arithmetical sense (*Algebra*, p. 62.) For instance, calling $k\theta$ the logarithm of $\varepsilon^{k\theta}$, we have, making $\theta = 2n\pi$, $n$ being a whole number ($\cos\theta = 1$, $\sin\theta = 0$)

$$\varepsilon^{2n\pi\sqrt{-1}} = 1 \qquad \log 1 = 2n\pi\sqrt{-1}$$

thus 1 has, under our present extension, an infinite number of logarithms, corresponding to all whole values of $n$, positive and negative, namely,

$$\ldots\ldots -2\pi\sqrt{-1}, \quad -\pi\sqrt{-1}, \quad 0, \quad \pi\sqrt{-1}, \quad 2\pi\sqrt{-1}, \ldots\ldots$$

among which the arithmetical logarithm is found, namely, 0. Similarly we may deduce

$$\varepsilon^{(2n+1)\pi\sqrt{-1}} = -1 \qquad \log(-1) = (2n+1)\pi\sqrt{-1}$$

Let $x$ be the arithmetical logarithm of $y$, then we have

$$y = \varepsilon^{x} = \varepsilon^{x}\times 1 = \varepsilon^{x} + \varepsilon^{2n\pi\sqrt{-1}} = \varepsilon^{x+2n\pi\sqrt{-1}}$$

or all the values of $x+2n\pi\sqrt{-1}$ are also logarithms of $y$. The theorem $\log a + \log b = \log ab$ now exists in this form: if any logarithm of $a$ be added to any logarithm of $b$, the sum is *one of the* logarithms of $ab$. Thus, if $\log. a$ stand for the arithmetical logarithm of $a$, and $\lambda a$ for its general logarithm, we have

$$\lambda a + \lambda b = \log a + 2\pi n\sqrt{-1} + \log b + 2\pi n'\sqrt{-1}$$
$$= \log(ab) + 2\pi(n+n')\sqrt{-1} = \lambda ab$$

(103.) We can now give the forms of all the different roots of a number, which was done to the fourth degree in p.113 of the *Algebra*. If we take the equation

$$1 = \varepsilon^{2n\pi\sqrt{-1}} = \cos 2n\pi + \sin 2n\pi\sqrt{-1}$$

$$\text{we have } (1)^{\frac{1}{m}} = \varepsilon^{2\frac{n}{m}\pi\sqrt{-1}} = \cos\frac{2n\pi}{m} + \sin\frac{2n\pi}{m}\sqrt{-1}$$

from which it might appear that there are as many *m*th roots of unity as there are whole values, positive and negative, of *n*. But it will be found that these values recur as we give to *n* different values in succession; as follows:

First root $n = 0$ First value of $(1)^{\frac{1}{m}}$ is $\cos 0 + \sin 0 . \sqrt{-1}$ or 1

Second root $n = 1$ Second .......... $\cos\frac{2\pi}{m} + \sin\frac{2\pi}{m}\sqrt{-1}$

Third root $n = 2$ Third .......... $\cos\frac{4\pi}{m} + \sin\frac{4\pi}{m}\sqrt{-1}$

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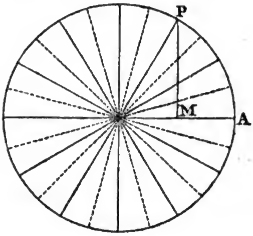
*m*th root $n = m - 1$ *m*th .......... $\cos\frac{2(m-1)\pi}{m} + \sin\frac{2(m-1)\pi}{m}\sqrt{-1}$

The $(m+1)$th value is $\cos\frac{2m\pi}{m} + \sin\frac{2m\pi}{m} . \sqrt{-1}$ or $\cos 0 + \sin 0 \sqrt{-1}$, the same as the first. The $(m+2)$th value is $\cos\frac{2(m+1)\pi}{m} + \sin\frac{2(m+1)\pi}{m}\sqrt{-1}$ or $\cos\left(2\pi + \frac{2\pi}{m}\right) + \sin\left(2\pi + \frac{2\pi}{m}\right)\sqrt{-1}$, the same as the second, and so on. To find the twelfth roots, for



instance, draw a circle, with the centre O. Divide its circumference into 12 equal parts, beginning at A, any given point. Then every point of subdivision shews a root as follows. If the radius be the linear unit, and if PM and MO represent the fractions of a linear unit which are in those lines; then,

$$\text{one value of } (1)^{\frac{1}{12}} \text{ is } \mathrm{OM} + \mathrm{PM}\,.\sqrt{-1}$$

(104.) It may be very soon shewn, that if $c + s\sqrt{-1}$ be a root of unity, $c - s\sqrt{-1}$ is also a root. Firstly, because if $\varepsilon^{x\sqrt{-1}}$ be a root, or if

$$\varepsilon^{nx\sqrt{-1}} = 1 \text{ we have } \left(\varepsilon^{nx\sqrt{-1}}\right)^{-1} = 1 = \left(\varepsilon^{-x\sqrt{-1}}\right)$$

or, if $\cos x + \sin x\,.\sqrt{-1}$ be a root, $\cos x - \sin x\sqrt{-1}$ is another. Secondly, in the preceding list of roots we see from

$$\frac{2(m-1)\pi}{m} \text{ or } 2\pi - \frac{2\pi}{m} \text{ that the last root is } \cos\left(-\frac{2\pi}{m}\right) + \sin\left(-\frac{2\pi}{m}\right)\sqrt{-1}$$

$$\text{or } \cos\frac{2\pi}{m} - \sin\frac{2\pi}{m}\sqrt{-1} \text{ the second being } \cos\frac{2\pi}{m} + \sin\frac{2\pi}{m}\sqrt{-1}$$

The same will also appear from consideration of the figure.

(105.) By proceeding in the same manner with $\varepsilon^{(2n+1)\pi\sqrt{-1}}$ or $-1$, we find

$$(-1)^{\frac{1}{m}} = \cos\frac{(2n+1)\pi}{m} + \sin\frac{(2n+1)\pi}{m}\sqrt{-1}$$

and, as before, it may be proved, that there are $m$ roots, and no more. But the roots of $-1$ may be more easily set forth to the eye, by means of the roots of $+1$, as follows. Every root of $-1$ is twice as high a root of $+1$; for if $x^m = -1$, then $x^{2m} = 1$. Consequently, if we take all the twenty-fourth roots of unity, or divide the circle in last article into 24 parts, all those 24th roots of $+1$ which are not also 12th roots, are 12th roots of $-1$.

(106.) Let all roots of unity of the same order be called corresponding roots: thus there are $m$ corresponding roots of $+1$ of the $m$th order. Then, all powers of a root are corresponding roots. For if $\mu^m = 1$, then $\mu^{2m}$ or $(\mu^2)^m = 1$, &c. And this holds equally of all negative whole powers. But it does not therefore follow that, among the powers of any one root, will be found all the other corresponding roots.

Let us denote $\varepsilon^{\pi x\sqrt{-1}}$ by $[x]$ for the present. Then we have

$$[nx] = [x]^n, \quad [x+2p] = [x], \quad [2p] = 1$$

where $n$ and $p$ are whole numbers, positive or negative. The $m$th roots of unity are

$$1 = \left[0\right], \ \left[\frac{2}{m}\right], \left[\frac{4}{m}\right], \left[\frac{6}{m}\right], \text{ up to } \left[\frac{2(m-1)}{m}\right].$$

Let us consider the case of $m = 8$. Then the powers of $[0]$ are severally $= [0]$, and never produce more than one root. The powers of $[2 \div 8]$ are as follows:

$$\left[\frac{2}{8}\right], \left[\frac{4}{8}\right], \left[\frac{6}{8}\right], \left[\frac{8}{8}\right] = -1, \ \left[\frac{10}{8}\right], \left[\frac{12}{8}\right], \left[\frac{14}{8}\right], \left[\frac{16}{8}\right] = 1, \ \&\text{c.}$$

or all the roots are produced in order. But the powers of $\left[\frac{4}{8}\right]$ are

$$\left[\frac{4}{8}\right], \left[\frac{8}{8}\right] = -1, \ \left[\frac{12}{8}\right], \left[\frac{16}{8}\right] = 1, \ \left[\frac{20}{8}\right] = \left[\frac{4}{8}\right] \ \&\text{c.}$$

and there is a continual recurrence of half the roots only. The powers of $\left[\frac{6}{8}\right]$ are

$$\left[\frac{6}{8}\right], \left[\frac{12}{8}\right], \left[\frac{18}{8}\right] = \left[\frac{2}{8}\right], \left[\frac{24}{8}\right] = -1,$$

$$\left[\frac{30}{8}\right] = \left[\frac{14}{8}\right], \ \left[\frac{36}{8}\right] = \left[\frac{4}{8}\right], \ \left[\frac{42}{8}\right] = \left[\frac{10}{8}\right],$$

$\left[\frac{48}{8}\right] = 1$, followed by recurrence.

Here again are all the roots. Those roots, whose powers give all the roots of their kind, are called *primitive*. It is enough for our present purpose to know that there is one primitive root of any order.

(107.) The roots of $+\sqrt{-1}$ and $-\sqrt{-1}$ may be now obtained. For we have

$$\varepsilon^{\frac{\pi}{2}\sqrt{-1}} = \sqrt{-1} = \varepsilon^{(2n+\frac{1}{2})\pi\sqrt{-1}}, \quad \varepsilon^{\frac{3\pi}{2}\sqrt{-1}} = -\sqrt{-1} = \varepsilon^{(2n+\frac{3}{2})\pi\sqrt{-1}}$$

$$(\sqrt{-1})^{\frac{1}{m}} = \varepsilon^{\frac{4n+1}{2m}\pi\sqrt{-1}} \qquad (-\sqrt{-1})^{\frac{1}{m}} = \varepsilon^{\frac{4n+3}{2m}\pi\sqrt{-1}}$$

which may, as before, be shewn to have $m$ values only.

To take an instance for verification: the cube roots of $+\sqrt{-1}$ are

$$\cos\frac{1}{6}\pi+\sin\frac{1}{6}\pi\sqrt{-1} \quad \cos\frac{5}{6}\pi+\sin\frac{5}{6}\pi\sqrt{-1} \quad \cos\frac{9}{6}\pi+\sin\frac{9}{6}\pi\sqrt{-1}$$

$$\text{or} \quad \frac{1}{2}\sqrt{3}+\frac{1}{2}\sqrt{-1} \quad -\frac{1}{2}\sqrt{3}+\frac{1}{2}\sqrt{-1} \quad -\sqrt{-1}$$

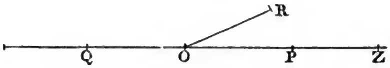
$$\left(\frac{1}{2}\sqrt{3}+\frac{1}{2}\sqrt{-1}\right)^3=\frac{1}{8}\left(3\sqrt{3}+9\sqrt{-1}-3\sqrt{3}-\sqrt{-1}\right)=\sqrt{-1}$$

$$(-\sqrt{-1})^3=\sqrt{-1}$$

(108.) Having shewn that the rules which *would* apply to $\sqrt{-1}$ *if* it could be considered as a quantity, will of themselves make all the necessary distinctions between the formulæ of common algebra, and the shape in which they become formulæ of trigonometry, we shall now proceed to the second method of explanation, or rather to the method of application, which shews the geometrical meaning of the symbols in question. As a starting point, we return again to the method of explanation of the negative sign. In looking at $a+(-b)$ we found that the sign $+$ no longer preserved the meaning of arithmetical addition, while the quantity operated on, $-b$, was no longer simply a number, but a number with a sign of direction. We might apply the preceding method of explanation to the passage from arithmetic to algebra. In this case $ka$ would signify a distinction of this kind; terms having $k^2$, $k^4$, &c. are all to be of one kind, unmarked, while terms having $k$, $k^3$, $k^5$, &c. are all to be of another kind, marked with the distinction $k$. If we adopted this signification we should soon find that all theorems which are true when the distinction $k$ means nothing, and may be entirely abolished, are also true when the distinction $k$ means simply *change of sign, if it be arithmetically allowable.* And it would also be found that were it allowable to consider $0-1$ as a quantity, the rules which would apply to this latter symbol are precisely those by which the necessary distinctions would be drawn in the course of the process, without any particular attention. We might thus dispense with subtraction at the outset, and establish all theorems in which arithmetically additive terms only occur. And subtraction might be introduced in time by means of a distinctive symbol. We cannot make this an illustration for a beginner, because to place it on the same footing as the subject of (73.), we must require him to imagine himself divested of some of his most simple notions, and thereby reduced to learn things which now are axioms, by a long process of reasoning. He must conceive himself unable to form a distinct notion of subtraction, other than the *inverse of addition;* that is, though the notion of taking away that which he himself has just added may be simple, yet the idea of instituting a subtraction independently of any previously expressed addition, must be one to be learnt with some difficulty. If this were the case, we could make the use of a distinctive symbol facilitate the acquirement of the general operation of subtraction; as it is, the last-mentioned process is one of which we have a clear idea, independently of addition.

(109.) We found that the interpretation of a negative quantity was a magnitude taken in precisely the opposite sense and meaning to that which we imagined, when we applied arithmetical process to the determination of that magnitude. And the affairs of life contain so many such interpretations that the extension looks natural; so natural indeed, that many have drawn a great distinction between the negative quantity, and the square root of the negative quantity. Of this the application of the term impossible to the latter symbol only, is a sufficient instance: both are impossible, according to arithmetical notions, but the latter only has received the name. There is hardly a phenomenon in nature, or a relation of life, which does not admit of the modes of neutrality, excess on one side, or on the other. Wherever there are two opposites, with a state between them which does not belong to either, we have the means of illustrating the terms, *positive*, *nothing*, and *negative*. Without specifying what we are speaking of, it would at once be granted that where the establishment of one state of things would cause increase, that of the opposite state would cause diminution, and that of the neutral state neither increase nor diminution. If we consider time as divided into *before*, *after*, and *at;* pecuniary relations divided into those of debtor, creditor, and neither debtor nor creditor; a balloon with weights, as either rising, sinking, and neither rising nor sinking; electricity as producing attraction, repulsion, or neither attraction nor repulsion, &c. &c. we have the same common modes of existence: *and in all the cases we have mentioned, no others whatsoever.* It is in considering *space* only that we have these modes, *and others,* as follows:

(110.) As long as we consider ourselves at liberty to change the position of a point in a given straight line, and in that straight line only, we have the three modes already considered, and no others. The point P may lie on one side or the other of a standard point O, given to measure from, or it may coincide with the point O. Let our notion of space be contained in length, and we pass with perfect continuity from P to Q, by continually diminishing OP, until P coincides with O, and then continuing the motion of P on the left. We have here perfect analogy with all the other kinds of quantity which we can conceive. Let O represent the commencement of the Christian era; take an inch to a year, and we have the means of making a table of all conceivable modes of time, and by laying down points corresponding to different events, we might reduce chronology to a science of feet and inches. *But* if we allow all space to enter the question, we may bring OP into the position OQ without diminishing its length, by turning it round through two right angles. And if we now consider the first passage from P to Q, relatively to the new notion we have introduced, we see direct *discontinuity*. The line OP always continues making an angle *nothing** with its first position, until P coincides with O, when OP is not a line, and the idea of opening, actual or possible, ceases altogether: immediately afterwards, OP, now become OQ, makes two right angles, or half a revolution, with its first position. But it might have made this at two steps of a right angle each, or at four of half a right angle each, &c. &c. Consequently, in geometry, we can pass from positive to negative by an infinite number of gradations, but which, as yet, we have no means of noting, though the conception is clearly attained.



Now let us try to extend our notion of chronology in the same manner. Let O denote the commencement of the Christian era; let years A.C be measured to the left in inches, and A.D to the right.

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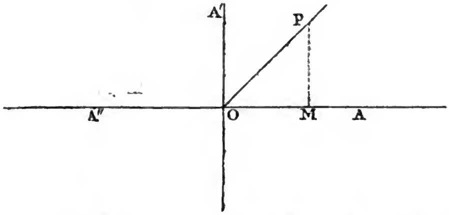
* The student must carefully distinguish between *no opening where opening might have been*, and *no opening because opening is inconceivable*. And this distinction must be made throughout the mathematics. Our language wants a phrase to distinguish between the nonexistence which arises from incoherence of ideas, and that which is not, but might have been, the case.

We can easily say what point of time P represents, or where P should be in order that it might represent any given event. But what is the answer to the question, What point of time does R denote? This question is wholly unanswerable. All the time we can form a notion of, is already expressed on one side or other of O; there remains no idea of time answering to the point R. We have no idea of time except *quantity* of duration: we have two ideas of a straight line, *quantity of length*, and *direction*.

(111.) We see then, that in preparing an algebra for geometry, we are making one which will be more than we can apply to any thing else. But we shall carry on our geometrical algebra, and then shew that, by a process of pure reasoning, which contains no assailable point,* we can use this method in all other sciences.

Let all lines be considered as having *direction* as well as *magnitude*, and both essential to their definition; so that two lines shall not be called equal, unless they be in fact equal and parallel. Lines of equal length in different directions are to have distinct symbols, and to be different things. Let there be only one direction of revolution (for the present), namely, from OA through OA′, a direction, OA, having been chosen which is to be permanently that of positive quantity, or that in which a line is to be expressed by its simple symbol $a$U, where U is the linear unit, and $a$ the number of linear units, or (if the student think he comprehends the phrase) the symbol for the ratio of the line to the unit. Consequently, a line measured in the direction OA″ has such a symbol as $-a$U, supposed to be already understood. Call OA .... the *line of arithmetic*, which is part



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* When I say this, I mean that all the objections which have been made relatively to negative and impossible quantities in the usual sense, right or wrong, have nothing to do with the reasoning which will be employed.

of .... A″OA .... that of the algebra of positive and negative quantities. *In this line the meaning of + and — is fixed by all that has preceded in ordinary algebra;* but as relates to lines not coinciding with OA or OA″, nothing must be conceived to have been yet laid down. The meaning of + and — remains to be determined, subjec only to the *law of extension*, that *all* the limited meaning is to be contained in the extended meaning. The question to be determined is, what is the proper representation of the line OP, $r$U in *length*, and $\theta$ in direction, meaning that it makes an angle $\theta$ with OA.

Firstly, suppose $\theta$ to be commensurable with two right angles, or that $\theta\Theta = \frac{m}{n}\pi\Theta$ ($m$ and $n$ whole numbers) whence $2n\theta = m.2\pi$.

Let $k^{\theta}r$ be the distinctive symbol which marks that $r$U makes an angle $\theta$ with the arithmetical line; where $k$ not being a symbol of quantity, neither is $\theta$ that of an exponent, in the algebraical sense. Then it is evident that, by turning OP until it has revolved through $2n$ times the angle $\theta$, we bring OP into the position of a line making $m$ sets of four right angles with OA, that is, we bring it into the direction OA for the $m$th time. And because coincidence is restored after every revolution, we must have

$k^{\theta+2m\pi}r\,\mathrm{U}$ signifying the same length and direction as $k^{\theta}r\,\mathrm{U}$

$k^{2m\pi}\mathrm{U}$ .................................. U

or $k^{\theta+2m\pi}r\,\mathrm{U}$ *equals* $k^{\theta}r\,\mathrm{U}$ $\qquad k^{2\pi m}\,\mathrm{U} = \mathrm{U}$

for we have said, let lines be *equal* when they are the same in length and direction.

If then we want such a symbol as is capable of expressing the necessary distinctions, and which shall be pointed out by the rules of ordinary algebra, we must so assign $k^{\theta}$ as that $k^{\theta}.k^{\theta}.k^{\theta}....(2n) = k^{2\pi}\,k^{2\pi}....(m) = 1$, which can be done by making $k^{\theta}$ a $2n$th root of 1, in the manner already laid down, where $\sqrt{-1}$ is treated* as a quantity. If we ask which root of 1 is to be adopted, the answer evidently is, that root which, when raised to the $2n$th power, gives the unit at the end of its $m$th revolution, or exhibits it in the form $\cos 2m\pi + \sin 2m\pi . \sqrt{-1}$. That is, we must let

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* Let the student carefully note the difference between *treating* $\sqrt{-1}$ as a quantity, when it has been proved that certain purposes of distinction are answered by so doing, and *considering* it as a quantity. In *operations*, this difference amounts to nothing, which is precisely the reason for which we adopt the method; but in *theory*, the first method gives *good reasoning*; the second gives only a part of it, *logical deduction*.

$$k^{\theta} \text{ be signified by } \cos\frac{2m\pi}{2n} + \sin\frac{2m\pi}{2n}. \sqrt{-1}$$

(Be careful to remember that $\sin x\sqrt{-1}$ means, *not* $\sin(x\sqrt{-1})$ *but* $(\sin x).\sqrt{-1}$). By De Moivre's Theorem,

$$\left(\cos\frac{2m\pi}{2n} + \sin\frac{2m\pi}{2n}\right)^{2n} = \cos 2m\pi + \sin 2m\pi \sqrt{-1}$$

a form of unity.

But $\frac{2m\pi}{2n} = \frac{m}{n}\pi = \theta$; hence $k^{\theta}$ is $\cos\theta + \sin\theta\sqrt{-1}$

Secondly, let $\theta$ and $\pi$ be incommensurable; the same notation must still be preserved, extending to the symbol of quantity $\theta$ all those considerations which have been heretofore introduced, in respect to the connexion of commensurables and incommensurables. The general result then is,

$r(\cos\theta + \sin\theta\sqrt{-1})$ U signifies a line $r$U inclined at an angle $\theta$ to the arithmetical line.

(112.) The meaning of the sign + before a term distinguished by $\sqrt{-1}$, is to be determined from the preceding. If OM = $x$U (= is here correctly applied), and if MP equals in length $y$U (the limited definition of =), we find that the line is

$$x + y\sqrt{-1};$$

consequently, $x + y\sqrt{-1}$, or OM + MP$\sqrt{-1}$, is the symbol for OP; whence we see that our extension of meaning is as follows: whereas, in the arithmetical line, OM + MP (MP being carried forward on that line) would have been OP on that line, then OM + MP$\sqrt{-1}$, when MP is distinguished as being at right angles to OM, *still means* OP. That $\sqrt{-1}$ is the distinction of *perpendicularity*, as $-1$ is of *contrariety of direction*, appears as follows: the unit perpendicular to OA is U$\left(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\sqrt{-1}\right)$ or U$\sqrt{-1}$; but more evidently from the consideration that if $k$U denote perpendicularity, $k(k\text{U})$ must denote motion through a right angle more, or $k^2$U must denote $-1$.U.

(113.) Our extension is now complete: it is yet for us to see what extension the preceding makes in the remaining notions of algebra.

1. What is $k^{\theta}r + k^{\theta'}r'$. Let $x$ and $x'$ be values of OM, and $y\sqrt{-1}$, and $y'\sqrt{-1}$ those of MP for the lines just stated. Then we have

$$k^{\theta}r + k^{\theta'}r' = x + x' + (y + y')\sqrt{-1}$$

Now, the general form $x + y\sqrt{-1}$ may be converted as follows:

$$x + y\sqrt{-1} = \sqrt{x^2 + y^2}\,(\cos\theta + \sin\theta\sqrt{-1})$$

where $\tan\theta = \frac{y}{x}$. This follows from the treatment of $\sqrt{-1}$ as a quantity; for if we assume

$$r\cos\theta = x \quad r\sin\theta = y; \text{ then } r = \sqrt{x^2 + y^2} \quad \tan\theta = \frac{y}{x}.$$

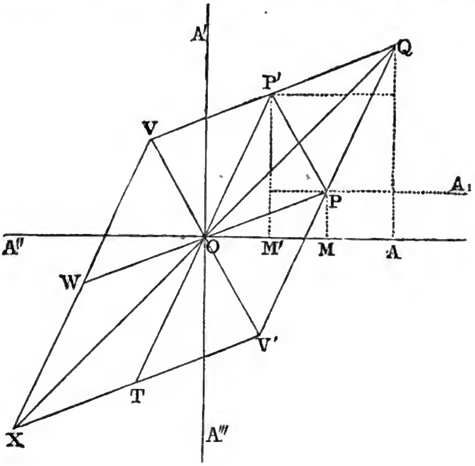
$$x + y\sqrt{-1} = r\cos\theta + r\sin\theta\sqrt{-1} = \sqrt{x^2 + y^2}(\cos\theta + \sin\theta\sqrt{-1})$$

Applying this to $x + x' + (y + y')\sqrt{-1}$, and making also $x' = r'\cos\theta'$ $y' = r'\sin\theta'$, we find

$$\sqrt{(x + x')^2 + (y + y')^2}\ (\cos\phi + \sin\phi\sqrt{-1}),\ \tan\phi = \frac{y + y'}{x + x'}$$

$$\sqrt{x^2 + y^2 + x'^2 + y'^2 + 2\,(xx' + yy')}\ (\cos\phi + \sin\phi\sqrt{-1}), \text{ or}$$

$$k\theta r + k^{\theta'}r' = \sqrt{r^2 + r'^2 + 2rr'\cos(\theta - \theta')}\,(\cos\phi + \sin\phi\sqrt{-1})$$



$= \mathrm{OQ}\,(\cos \mathrm{QOA} + \sin \mathrm{QOA}\sqrt{-1})$ since $\tan \mathrm{QOA} = \frac{\mathrm{QA}}{\mathrm{AO}} = \frac{y+y'}{x+x'}$: or the sum of OP and OP′ is OQ, the diagonal of the completed parallelogram, which ends at O. The difference of the two lines will be found in the same way to be

$$x - x' + (y-y')\sqrt{-1} = \sqrt{r^2 + r'^2 - 2rr'\cos(\theta-\theta')}\,(\cos\phi' + \sin\phi'\sqrt{-1})$$

$$(\text{where } \tan\phi' = \frac{y-y'}{x-x'} = \tan \mathrm{P'PA_1})$$

$$= \mathrm{PP'}(\cos \mathrm{P'PA_1} + \sin \mathrm{P'PA_1}\sqrt{-1})$$

or the difference $k^{\theta} r - k^{\theta'} r'$ is PP′, the other *diagonal*. Let $\overline{\mathrm{OP}}$ signify the line OP in *length and direction*. The way of settling the distinction between $\overline{\mathrm{OP}} - \overline{\mathrm{OP'}}$ and $\overline{\mathrm{OP'}} - \overline{\mathrm{OP}}$, is as follows. The opposite of any line is found by merely changing the sign of $r$, if we allow $\theta$ to remain the same. For the opposite of $r\cos\theta + r\sin\theta\sqrt{-1}$ is $r\cos(\theta+\pi) + r\sin(\theta+\pi)\sqrt{-1} = (-r)\,(\cos\theta + \sin\theta\sqrt{-1})$: that is, two opposite lines may be expressed, either by changing the sign of the symbol of length, or adding two right angles to that of direction. We see then that the opposite of $r(\cos\theta + \sin\theta\sqrt{-1})$ may either be defined as $r$ at the angle $\theta + \pi$, or as $-r$ at the angle $\theta$. Now, to determine the proper interpretation of $\overline{\mathrm{OP}} - \overline{\mathrm{OP'}}$, since we are to have all algebraical formulæ remain true, let us write it thus: $\overline{\mathrm{OP}} + (-\overline{\mathrm{OP'}})$, and add $\overline{\mathrm{OP}}$ and $-\overline{\mathrm{OP'}}$, or $\overline{\mathrm{OP}}$ and $\overline{\mathrm{OT}}$; the result is $\overline{\mathrm{OV'}}$. Similarly $\overline{\mathrm{OP'}} - \overline{\mathrm{OP}} = \overline{\mathrm{OP'}} + (-\overline{\mathrm{OP}}) = \overline{\mathrm{OP'}} + \overline{\mathrm{OW}} = \overline{\mathrm{OV}}$. In truth, when we came to the first square root, namely, that which gave OQ, we should have ascertained that the sum was $\overline{\mathrm{OQ}}$ and not $\overline{\mathrm{OX}}$. This must be, for the extended definitions are entirely to contain the limited ones. Let OP and OP′ revolve towards the arithmetical line, and it is finally OQ, which becomes their arithmetical sum, and not OX.

(114.) The terms *greater* and *less* cannot have meaning as applied to lines defined in length and direction. We may have lines greater in length, or greater in direction, than others; but $\overline{\mathrm{OP}}$ is not greater or less than $\overline{\mathrm{OP'}}$. Watching this more narrowly, we see that we have defined *equal* in a sense which only applies to lines in the same direction; this *limitation*, for such it is, requires a corresponding limitation of the relative terms, greater and less.

(115.) As to multiplication, we see that

$$r(\cos\theta + \sin\theta\sqrt{-1}) \times r'(\cos\theta' + \sin\theta\sqrt{-1}) =$$
$$rr'(\cos(\theta+\theta') + \sin(\theta+\theta')\sqrt{-1})$$

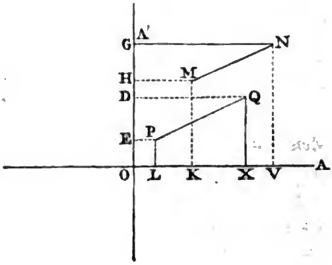
or, in multiplication of $r$ and $r'$, the result belongs to a line $rr'$U, with a direction, the sum of the directions of $r$ and $r'$. Similarly, $\frac{r}{r'}$, belongs to a line $\frac{r}{r'}$U, at an angle $\theta - \theta'$, and so on. We see then that the angles of direction have the properties of logarithms relatively to the lines, which is also plainly shewn by (or, if not, is a confirmation of)

$$r(\cos\theta + \sin\theta\sqrt{-1}) = r\varepsilon^{\theta\sqrt{-1}}$$

Taking this convenient abbreviation, we have a set of equations, as follows:

$$r\varepsilon^{\theta\sqrt{-1}} \times r'\varepsilon^{\theta'\sqrt{-1}} = rr'\varepsilon^{(\theta+\theta')\sqrt{-1}},$$
$$a\varepsilon^{\alpha\sqrt{-1}} \times b\varepsilon^{\beta\sqrt{-1}} \div c\varepsilon^{\gamma\sqrt{-1}} = \frac{ab}{c}\varepsilon^{(\alpha+\beta-\gamma)\sqrt{-1}},$$
$$\left(r\varepsilon^{\theta\sqrt{-1}}\right)^m = r^m\varepsilon^{m\theta\sqrt{-1}} \quad \left(r\varepsilon^{\theta\sqrt{-1}}\right)^{\frac{m}{n}} = r^{\frac{m}{n}}\varepsilon^{\frac{m}{n}\theta\sqrt{-1}}$$

(116.) If we look at our first method of introducing $\sqrt{-1}$, (99.) we see that the one just explained agrees with it, except in one important particular. That first method did not bind us to any meaning of the sign + and — before $k$, but left us, as we then might suppose, to imagine that P + $k$Q must be the same as P + Q, with a distinctive mark on Q, reserving it for future operations. But I did not notice at the time, wishing to avoid any generality which was not required by the subject, that the definition we assigned to $k$ entirely destroyed all specific meaning for the sign + before $k$, though it continued that meaning in relations among the terms independent of $k$, or among those affected with it. For if we agree to mean by P + $k$Q = P′ + $k$Q′ an equivalent to laying down both that P = P′ and Q = Q′, the preceding relation remains equally true, whatever may be the meaning of + before $k$; since identical operations performed upon equal quantities must give the same result. In applying the result to geometry, we find ourselves led to a more special meaning of $k$, for $k$Q means the line Q perpendicular to the arithmetical line. And with it we find a special meaning of + between a term without $k$ and one with $k$, namely, the hypothenuse of a right-angled triangle. But still the meaning of $k$ remains in force, for, in the geometrical application, $P = P'$ and $Q = Q'$. By expressly defining two lines as equal which have both equal lengths and directions, we require also equal projections on both OA and OA′, as is readily proved from the accompanying figure. For, if MN and PQ have



the same length and direction, then $KV = LX$, and $GH = DE$. But we have

$$\overline{MN} = KV + GH\sqrt{-1}$$
$$\overline{PQ} = LX + DE\sqrt{-1}$$

(117.) We have thus converted every theorem of algebra into one of geometry, not belonging to a class very useful at present, but, geometrically considered, of great complexity. And we have thus the satisfaction of putting upon every theorem of algebra a meaning which is as intelligible as an arithmetical operation, when the latter is as complicated; and this on the supposition that every letter is a (till now) impossible binomial of the form $x + y\sqrt{-1}$. Thus, in $a + b + c = c + b + a = b + a + c$, &c. we see the following theorem. If there be any number of straight lines meeting in a point, and if the conterminous diagonal of the parallelogram formed by any two be made one side of a new parallelogram, and a third line another; and if the conterminous diagonal of the last be similarly combined with a fourth line, and so on till all the lines are exhausted; then shall the last found diagonal be the same in magnitude and position, in what order soever the several lines were introduced. But theorems which are algebraically very simple lead to problems of great complexity. To shew this we shall enunciate the geometrical theorem which answers to

$$(a+b)^2 = a^2 + 2ab + b^2$$

as an exercise for the student in the meaning of the extension.

If there be two finite straight lines making angles with a third finite line on the same side, and all meeting in one point, and if we take the third proportional to the third line and each of the two first, and incline such third proportionals to the third line at angles twice as great as the first and second line; and if we also take the double of a fourth proportional to the third line and the two first and incline it to the third at an angle as great as the angles of the first and second together; and if we then take the conterminous diagonal of the parallelogram whose sides are the two third proportionals, and also the diagonal of the parallelogram which has the last-mentioned diagonal, and the double of the fourth proportional for its sides: then shall this last-mentioned diagonal be in length a third proportional to the third line and the diagonal of the parallelogram on the first two, and shall make, with the third line, an angle double of the angle made by the diagonal of the first two with the third.

The student should *construct* some problems of this kind, both as an exercise in the meaning of the extensions, and the use of the instruments.

(118.) I now pass to a question of much greater importance, namely, the use of the preceding extensions in reasoning. As a certain species of fallacy is called reasoning in a circle, I think the method I am going to describe might be called reasoning in a triangle; for, when there is an obstacle in one side, we pass from one end of it to the other over the other two, which we shall shew are free. To return to our illustration; suppose a question involving *times* before and after a certain epoch to be given, and suppose that ordinary algebraical reasoning, though it produce an answer positive or negative, does it by means of square roots of negative quantities fallaciously entering in the process. I say fallaciously (110.), because we cannot extend to our notions of duration those relations which answer to various directions of lines, except only, that if time before may be represented by a line north, time after may be represented by a line south. But a line east has no correlative in our notions of time. As long as we keep the notion of time, we must, so to speak, keep in the algebraical line (111.) of positive and negative quantity. But let us proceed as follows. Whatever ratios exist between the given times are made to exist between the geometrical lines which represent them in .... A″OA .... ; and if there be an answer to the question *in time*, there is an answer in the line A″OA to a given geometrical problem, corresponding to the given question. And, conversely, if there be in the line A″OA an answer to the geometrical question, namely, a line whose length has all the required ratios to the given lines, there must be a time whose ratio has all the required ratios to the given times; for ratios are the same things whether they be of portions of duration or of length. We cannot reason on the problem throughout when the concrete magnitudes are times: for, by our supposition, modes of duration of which we cannot conceive the existence are introduced. But we can reason on the geometrical problem, because geometry can put an intelligible construction upon correlative modes which exist among lines in different directions. But the geometrical result when obtained, gives an answer to the problem upon the relations of time; not depending upon the methods which gave the geometrical answer in any way, but upon a circumstance altogether different; namely, that relations among lines, positive or negative, however obtained, have their correlative relations among times. That is to say, we may depend upon the results of *general* algebra, even when the concrete magnitudes under discussion are such as do not admit of the geometrical extension. But, if the geometrical problem give an answer which amounts to supposing a line OP not in the line of algebraical positives or negatives, then we know that the corresponding problems relative to such concretes as do not admit of the extension, are strictly impossible, and (as yet, at least) inconceivable in any sense whatsoever. The same reasoning applies if the given problem be upon abstract numbers, with this additional limitation, that the question is impossible unless the answer of the corresponding geometrical problem lie upon the *arithmetical* line OA ...

We shall not in future consider it necessary to make any distinction between $\sqrt{-1}$ and other symbols. Those equations in which it occurs are rationally true in the extended sense of the symbols, and those in which it does not occur are true in the ordinary algebraical sense; for, as seen, the extended and ordinary meanings coincide when the symbol $\sqrt{-1}$ is neither expressed nor implied.

# CHAPTER V.

## ON THE DISTINCTION OF *PERIODIC* QUANTITIES, AND PRELIMINARY CONSIDERATIONS ON THE INVERSION OF PERIODIC FUNCTIONS.

(119.) We have, in the notion of continual revolution, an idea of quantity, the whole effect of which is represented by the angle which the revolving line makes with the line of commencement; and in which, so far as position is concerned, or any ratios which depend upon position, it is indifferent whether the place which the line occupies be in the first, second, or any other revolution. If we were to say, let the angle $2\pi + \theta$ be the same angle as $\theta$, we should then call an angle a *periodic* quantity; but this notion would not be very accurate. Yet $\sin\theta$, $\cos\theta$, &c. are really *periodic* functions of $\theta$, for as $\theta$ increases they do not increase or decrease without end, but circulate in value through various changes, in such manner that what value soever any one has when $\theta = a$, it has the same for $a + 2\pi$, $a + 4\pi$, &c. It is usual to distinguish the primary functions of angles from magnitudes which depend on lines, or other continually increasing quantities, by the name of *periodic* quantities.

(120.) The question of finding the double, treble, &c. of a quantity of revolution is not affected by the multiplicity of values which answer to that quantity, considered as the determiner of angular position. Thus $m\theta$, $m(2\pi+\theta)$, $m(4\pi+\theta)$, &c. are $m\theta$, $2m\pi+m\theta$, $4m\pi+m\theta$, which all give the revolving line the same position as $m\theta$, when $m$ is a whole number. But when $m$ is a fraction, the case is altered. Suppose we ask, how many different quantities of revolution are there, each of which repeated 10 times, will leave the revolving line at an angle $\theta$ with the primary line OA. This position may be obtained by the revolutions $\theta$, $\theta + 2\pi$, $\theta + 4\pi$, &c. consequently, the tenth part, or the solution necessary, is contained in the formula

$$\frac{2m\pi+\theta}{10} \quad \text{or} \quad \frac{2m}{10}.\,\pi + \frac{\theta}{10}, \text{ the values of which are}$$

$$\frac{\theta}{10},\ \frac{\pi}{5}+\frac{\theta}{10},\ \frac{2\pi}{5}+\frac{\theta}{10}\ \ldots\ \text{up to}\ \frac{9\pi}{5}+\frac{\theta}{10},$$

after which there is only recurrence; for $\frac{10\pi}{5}+\frac{\theta}{10}$ is $2\pi+\frac{\theta}{10}$. If then, we were to set out with a problem involving the given angle $\theta$, and the answer were,

$$\text{the value required is } \cos\frac{\theta}{10}$$

there would, in fact, be ten answers; for $\theta$ might at the outset have been called $2\pi+\theta$, or $4\pi+\theta$, &c.; and though these angles have equal cosines, their tenth parts have not.

Thus, in the formula $\sin\theta = 2\sin\frac{1}{2}\theta.\cos\frac{1}{2}\theta$, we see that

$$\sin\theta \text{ is either } 2\sin\tfrac{1}{2}\theta.\cos\tfrac{1}{2}\theta \text{ or } 2\sin(\pi+\tfrac{1}{2}\theta)\cos(\pi+\tfrac{1}{2}\theta)$$

the latter arising from writing $2\pi+\theta$ for $\theta$. And in

$$\cos\tfrac{1}{2}\theta+\sqrt{-1}\sin\tfrac{1}{2}\theta = \pm\sqrt{\cos\theta+\sqrt{-1}\sin\theta}$$

the second side has two values; and so has the first side, its second value appearing by writing $2\pi+\theta$ for $\theta$.

(121.) The actually periodic character of the series in (101.) is a point of interest, which it may be advisable to verify. An angle is expressed in analytical units by an absolute number, and every number belongs to some quantity of revolution, 1 belonging to that which makes the arc and radius equal. And the remark in (63.) must be particularly attended to. No result yet obtained, which involves angles themselves, is true of degrees, minutes, and seconds, but only of analytical units. If we take a high number for $\theta$, the formation of a few terms will make the series appear very divergent; but we see that convergence must come, for, in the sine, we have

$$(n+1)\text{th term} = n\text{th term} \times \frac{\theta^2}{2n(2n+1)}$$

And in the cosine

$$(n+1)\text{th term} = n\text{th term} \times \frac{\theta^2}{(2n-1)2n}$$

So that, however great $\theta$ may be, terms must arrive at which $2n(2n+1)$ and $(2n-1)2n$ bear as great a ratio as we please to $\theta^2$, or the $(n+1)$th terms may be made parts as small as we please of the $n$th. And the terms being alternatively positive and negative, the magnitude of the latter kind will compensate that of the former, leaving a difference always less than unity, as should be the case from (36.). To shew this, we shall take some terms of the sine and cosine of the angle 10, something greater than $3\pi$. We begin with 10, and the process is continual multiplication by 10, and division by successive numbers. Let $(n)$ mean $10^n$ divided by $1.2.3 \ldots . n$, then we have

$$\begin{aligned} \sin 10 &= (1) - (3) + (5) - (7) + \cdots\cdot \\ \cos 10 &= \;\,1\; - (2) + (4) - (6) + \cdots\cdot \end{aligned}$$

To pass from $(n)$ to $(n+1)$ we must multiply by 10 (move the decimal point one place to the right) and divide by $n+1$. To get a sine and cosine from this (at the beginning) very diverging series, to a single place of decimals, will require about 29 places of decimals to be considered in the first term. Let us begin with

$$\begin{aligned} (1) &= 10{\cdot}0000 \ldots . \text{ 29 ciphers} \\ (2) &= 50{\cdot}000 \ldots . \\ (3) &= 166{\cdot}66 \ldots . \;\; \&\text{c.} \end{aligned}$$

Proceeding in this way, and keeping two decimals from each term, we find as follows:

| | | | | | |
|---|---|---|---|---|---|
| (1) = | 10·00 | (11) = | 2505·21 | (21) = | 19·57 |
| (2) = | 50·00 | (12) = | 2087·68 | (22) = | 8·90 |
| (3) = | 166·67 | (13) = | 1605·90 | (23) = | 3·87 |
| (4) = | 416·67 | (14) = | 1147·07 | (24) = | 1·61 |
| (5) = | 833.33 | (15) = | 764·72 | (25) = | ·64 |
| (6) = | 1388·89 | (16) = | 477·95 | (26) = | ·25 |
| (7) = | 1984·13 | (17) = | 281·15 | (27) = | ·09 |
| (8) = | 2480·16 | (18) = | 156·19 | (28) = | ·03 |
| (9) = | 2755·73 | (19) = | 82·21 | (29) = | ·01 |
| (10) = | 2755·73 | (20) = | 41·10 | | |

Hence, $(1) + (5) + (9) + \ldots .$ up to $(29) = 5506{\cdot}33 = \text{A}$

$(3) + (7) + (11) + \ldots .$ up to $(27) = 5506{\cdot}90 = \text{B}$

$\sin 10 = \text{A} - \text{B}$ nearly $= -{\cdot}57$

$1 + (4) + (8) + \ldots .$ up to $(28) = 5506{\cdot}20 = \text{C}$

$(2) + (6) + (10) + \ldots .$ up to $(26) = 5507{\cdot}03 = \text{D}$

$\cos 10 = \text{C} - \text{D}$ nearly $= -{\cdot}83$

$10 = 3\pi + {\cdot}58$ nearly, or $\cos 10 = -\cos{\cdot}58 \quad \sin 10 = -\sin{\cdot}58$

${\cdot}58\,\theta = {\cdot}58 \times 57{\cdot}31^\circ = 33{\cdot}23$ nearly *in degrees.*

And from the tables $\sin 55°\tfrac{1}{4} = \cdot 54$, $\cos 55°\tfrac{1}{4} = \cdot 84$, whence the first decimal is right, as we proposed. We have thus shewn how to verify the actual coincidence of the series with the sine and cosine of the tables.

(122.) When an operation is performed successively upon a quantity and its results, if we denote the operation by $f$, the results of the repeated operations may be denoted by $ff$, $fff$, &c., which may be abbreviated into $f^2$, $f^3$, &c. So that suppose $x$ the quantity operated upon, and $2x+1$ the operation to be performed (that is, let $f$ represent a direction to double, and then add one), we have

$$fx = 2x+1, \quad f^2x = 2(2x+1)+1 = 4x+3,$$
$$f^3x = 2\{2(2x+1)+1\}+1 = 8x+7, \text{ \&c.}$$

The convenience of this notation consists in the analogy which exists between the indices of $f$ and algebraical exponents. Thus we have

$$f^2f^2x = f^4x = 2\left[2\{2(2x+1)+1\}+1\right]+1;$$

the only difference being, that in $f^2f^2$ we consider the preceding as having all between $\{\}$ first finished, which total operation is then repeated; and in $f^4x$ we consider all the operations as separate. But both results, when developed, are, in fact, $ffffx$. To preserve the same equation, we must let $f^0x$ signify $x$, in order that

$$ff^0x \text{ may be } f^{1+0}x, \text{ or } f^1x, \text{ or } fx;$$

and the meaning of $f^{-1}x$ may thus be given. Let it be such that the equation $f^{m+n}x = f^m(f^nx)$ still exists, and we must then have

$$ff^{-1}x = f^{1-1}x = f^0x = x;$$

or $f^{-1}x$ means the inverse of $x$, the function whose effect is reversed or undone by $f$; so that if $f^{-1}$ be first performed upon $x$, and then $f$, the latter restores $x$ again. Thus, if $fx$ be $x^2$, $f^{-1}x$ is $\sqrt{x}$, because $ff^{-1}x$ or $(\sqrt{x})^2$ is $x$.

Since $f^{m+n}x$ is either $f^mf^nx$, or $f^nf^mx$, to preserve this equation, we should stipulate that $f^{-1}fx$ should be the same as $ff^{-1}x$, or that we should only allow functions satisfying this condition to be called $f^{-1}x$. Thus in $x^2$ and $\sqrt{x}$, we see that we have

$$(\sqrt{x})^2 = x, \text{ and, changing order of operation, } \sqrt{x^2} = x.$$

But if we take $-\sqrt{x}$ for $f^{-1}x$, $fx$ being $x^2$, we find

$$(-\sqrt{x})^2 = x, \quad -\sqrt{x^2} = -x, \quad f^{-1}fx, \textit{ not } \text{being} = x.$$

To elucidate this point, we must observe that, in algebra, direct operations, which produce but *one value* out of *one value*, are generally accompanied by inverses which produce more values than one; addition and multiplication only excepted. What do we mean by a square root? The expression, *which squared*, gives the original. This, with regard to $x^2$, answers both to $x$ and $-x$. But if we had been considering $x^2$ as an operation to be repeated, giving $(x^2)^2$, &c. and had from thence, in the preceding manner, deduced an idea of an inverse operation, we should have found that if

$$fx = x^2, \text{ and if } ff^{-1}x \text{ and } f^{-1}fx, \text{ are both to be } = x,$$

we can only admit $f^{-1}x = \sqrt{x}$, and not $f^{-1}x = -\sqrt{x}$. But because it has been customary in algebra to admit all functions to the name of *inverses* of $f$, which satisfy $ff^{-1}x = x$, without inquiring whether they satisfy $f^{-1}fx = x$, we must here adopt the name, but with a distinction of notation. Let those forms be called *convertible* inverse functions of $f$, and be denoted by $f^{-1}$, which satisfy both $ff^{-1}x = x$ and $f^{-1}fx = x$. And let those forms be called *inconvertible* inverse functions of $f$, and be denoted by $f_{-1}x$, which satisfy only $ff_{-1}x = x$, and *not* $f_{-1}fx = x$. Thus, when $fx = x^2$, $f^{-1}x = \sqrt{x}$, $f_{-1}x = -\sqrt{x}$. But it is right to warn the student that, in other works, $f^{-1}$ and $f_{-1}$ are not distinguished.

(123.) Let us now turn to $\cos x$. What is its inverse function? Let $\cos x$ be denoted by $fx$; then

$$\varepsilon^{x\sqrt{-1}} = \cos x + \sqrt{1-\cos^2 x}\,\sqrt{-1} = fx + \sqrt{1-(fx)^2}\,\sqrt{-1},$$

(as in my *Algebra*, p. 123, I always mean by $\sqrt{x}$ the positive square root, and by $x^{\frac{1}{2}}$ the general form, either positive or negative). This is true only for angles less than $\pi$ (of all in the first revolution); for if $x$ be greater than $\pi$, we must have

$$\varepsilon^{x\sqrt{-1}} = \cos x + \sin x\,\sqrt{-1},$$

and the second term (44.) is negative;

whence $$\varepsilon^{x\sqrt{-1}} = fx + \sqrt{1-(fx)^2}\,\sqrt{-1},$$

($x$ between 0 and $\pi$, $2\pi$ and $3\pi$, &c.)

$$\varepsilon^{x\sqrt{-1}} = fx - \sqrt{1-(fx)^2}\,\sqrt{-1},$$

($x$ between $\pi$ and $2\pi$, $3\pi$ and $4\pi$, &c.)

both included in $$\varepsilon^{x\sqrt{-1}} = fx + \left(1-(fx)^2\right)^{\frac{1}{2}}\sqrt{-1},$$

(for all values of $x$.)

Now, if for $x$ we write $x+2n\pi$, $fx$ remains as before, and so does $\varepsilon^{x\sqrt{-1}}$: we have then every possible form of the preceding equation in

$$\varepsilon^{x\sqrt{-1}+2n\pi\sqrt{-1}} = f(x+2n\pi)+\left\{1-(f\overline{x+2n\pi})^2\right\}^{\frac{1}{2}}\sqrt{-1}; \ldots (A)$$

in which there is identity of value in both sides of the equation for all values of $n$. But before we proceed a step further, we have an important remark to make, the neglect of which for a long time embarassed this subject.

(124.) *Identity of form* means positive and absolute identity in every respect. We see it in

$$x = x, \quad x+a = x+a, \quad \cos x = \cos x.$$

*Identity of value* includes all other cases in which the sign $=$ applies. We see it in

$$x = x+a-a, \ (b-a)^2 = (a-b)^2, \ \cos(x+2\pi) = \cos x.$$

*All operations, general or limited, performed upon identities of form,* produce the same results. We see this in

$$\sqrt{x} = \sqrt{x} \text{ as well as } x^{\frac{1}{2}} = x^{\frac{1}{2}}.$$

This proposition is only worth stating as a contradictory of the next.

*Limited operations performed upon identities of value with differences of form, do not necessarily produce the same results.* Thus,

$$(b-a)^2 = (a-b)^2 \text{ does not give } \sqrt{(b-a)^2} = \sqrt{(a-b)^2},$$

$$\text{or } b-a = a-b,$$

but it does give $$\left\{(b-a)^2\right\}^{\frac{1}{2}} = \left\{(a-b)^2\right\}^{\frac{1}{2}},$$

that is, *every value of the first side is* a value *of the second side, but not* *at our pleasure.* It is $\sqrt{x}$ on the first side, which is equal to the $-\sqrt{x}$ of the second, and *vice versá:* if we attempt to displace this arrangement, and make $\sqrt{x}$ of the first side equal to $\sqrt{x}$ of the second, we have no longer an *identity*, but an *equation of condition* (*Algebra*, p. ix.), for then $b-a = a-b$, or $a = b$.

(125.) Conversely, when we find that an operation performed upon both sides of an identity of value gives different results, it will be most judicious, before proceeding further, to examine first the operation in question, to see whether it be not a limited case of a more general operation. And we see that an algebraical equation, of which the sides admit of different values, such as arises from performing a general operation upon a general form, may be considered as an ambiguity of this form,

$$A_1, \text{ or } A_2, \text{ or } A_3 \ldots\ldots = B_1, \text{ or } B_2, \text{ or } B_3 \ldots\ldots$$

in which, though we know that $A_1$ has its equal on the other side, we have no right to say that it is $B_1$, for it may be any other. And the same of $A_2$, &c. And all we know as yet, and indeed all we shall find, will shew us that, in operations conducted in all their generality, there will be complete identity of this kind: every value of the first side will have its value on the other. We shall not have $A_1$ and $A_2$ both equal to $B_1$, and $B_2$ left without a value.

As long as our inversions only involved the production of two values, as in the square root, or of three, as in the cube root, there was little need to embarrass the subject by general ideas upon inversion: each case was the best index to its own peculiarities. But now, when the simplest cases we have to consider give infinite numbers of inversions (seeing, for example, that though an angle has but one cosine, a cosine has an infinite number of angles), we must watch our processes very narrowly. By trusting to limited notions of inversion, many errors have been introduced, some of which we shall point out in the sequel.

Returning to equation A, we must first find the arithmetical logarithm of the second side. We shall need some considerations, which the student will read with the more attention, when he knows that he is thus in reality entering upon a most essential part of his future studies, namely, the Differential and Integral Calculus.

## CHAPTER VI.

### NOTIONS OF THE DIFFERENTIAL AND INTEGRAL CALCULUS, SUCH AS WILL BE REQUISITE IN THE SUCCEEDING CHAPTERS.

(126.) LET any function of $x$, $fx$, have a certain value given to $x$, namely, $a$. Then let $x$ have the value $a+h$ given to it, and let the difference of the resulting values of the function be taken, namely, $f(a+h)-fa$. Instances:

$$\varepsilon^{a+h}-\varepsilon^{a},\ \sin(a+h)-\sin a,\ (a+h)^2-a^2.$$

This difference, if the function be continuous (*Algebra*, p. 102), is one which diminishes without limit at the same time as $h$. When the fraction $\dfrac{f(a+h)-fa}{h}$ (*Algebra*, p. 156) has a finite limit, let it be called the *derived* function of $fx$, for the value $x=a$. Or, in general, let the limit of $\dfrac{f(x+h)-fx}{h}$, made by diminishing $h$ without limit, be styled the *derived function* of $fx$, if it be a rational and continuous function of $x$. Let it be denoted by D, that is,

$$\text{Limit (if there be one) of } \frac{f(x+h)-fx}{h} = \mathrm{D}fx.$$

For example (*Algebra*, p. 225),

$$\frac{\varepsilon^{x+h}-\varepsilon^{x}}{h} = \varepsilon^{x}\frac{\varepsilon^{h}-1}{h} = \varepsilon^{x}\left(1+\frac{h}{2}+\frac{h^2}{2.3}+\ldots.\right)$$

Therefore (*Algebra*, p. 157.) $\mathrm{D}\varepsilon^{x} = \varepsilon^{x}$

$$\frac{\sin(x+h)-\sin x}{h} = \sin x\,\frac{\cos h-1}{h} + \cos x\,\frac{\sin h}{h} \text{ and by (46,47.)}$$

$$\mathrm{D}\sin x = \sin x\times 0+\cos x\times 1 = \cos x$$

$$\frac{\cos(x+h)-\cos x}{h} = \cos x\,\frac{\cos h-1}{h} - \sin x\,\frac{\sin h}{h} \quad (46,\ 47.)$$

$$\mathrm{D}\cos x = \cos x\times 0-\sin x\times 1 = -\sin x$$

(127.) Now precisely the same results are obtained if we use

$$\frac{f(x+h)-f(x+k)}{h-k}$$

and diminish $h$ and $k$ without limit. For instance,

$$\frac{\sin(x+h)-\sin(x+k)}{h-k} = \sin x\,\frac{\cos h-\cos k}{h-k} + \cos x\,\frac{\sin h-\sin k}{h-k}$$

$$= \sin x \times 2\left(-\frac{\sin\frac{1}{2}(h+k)\sin\frac{1}{2}(h-k)}{h-k}\right) + \cos x \times 2\,\frac{\sin\frac{1}{2}(h-k)\cos\frac{1}{2}(h+k)}{h-k}$$

$$= -\sin x \sin\tfrac{1}{2}(h+k)\,\frac{\sin\frac{1}{2}(h-k)}{\frac{1}{2}(h-k)} + \cos x \,.\, \cos\tfrac{1}{2}(h+k)\,\frac{\sin\frac{1}{2}(h-k)}{\frac{1}{2}(h-k)}$$

the limit of which, diminishing $h$ and $k$ without limit, is as already obtained,

$= \sin x \times 0 \times 1 + \cos x \times 1 \times 1$, or $\cos x$, that is, $\mathrm{D}\sin x$.

Let the student try the other instances in a similar manner.

(128.) Theorem. Required the relation which must exist between two functions which have the same derived function.

Let the derived function in question be P, and let Q and R be two functions, of both of which the derived function is P. Let $R = Q + T$, T being the difference of Q and R; and when $x$ is made $x + h$, let R′, Q′, and T′ be the values of the three, whence, since $R = Q + T$ is supposed to be an identical equation, we must have it true for all forms of $x$, and therefore we must have $R' = Q' + T'$; whence

$$\frac{R'-R}{h} = \frac{Q'-Q}{h} + \frac{T'-T}{h}.$$

Take the limits of both sides (*Algebra*, p. 156), diminishing $h$ without limit, and we have

$$\mathrm{D}R = \mathrm{D}Q + \mathrm{D}T, \quad \text{or} \quad \mathrm{D}T = 0,$$

because $\mathrm{D}R = \mathrm{D}Q = P$. Consequently T must be a function, whose derived function is nothing for all values of $x$. Now let $T(x)$ denote this function, where we make T the symbol of the functional operation, and express the quantity $x$, because we wish to express different values of it. Then let us suppose

$$\begin{aligned}
&T(x+h)-T(x) = V_1\\
&T(x+2h)-T(x+h) = V_2\\
&T(x+3h)-T(x+2h) = V_3\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&T(x+\overline{n-1}\,h)-T(x+\overline{n-2}\,h) = V_{n-1}\\
&T(x+nh)-T(x-\overline{n-1}\,h) = V_n
\end{aligned}$$

And by addition, $T(x+nh)-Tx = V_1+V_2+\ldots\ldots+V_n$.

Now, let us diminish $h$ without limit, increasing $n$ at the same time in such manner that $nh$ shall be always equal to a quantity *fixed in value, but what we please* (keep this in mind), and called $y$. (See the process in (101.)). By the nature of this derived function, $\frac{V_1}{h}, \frac{V_2}{h}$ .... must all diminish without limit with $h$, for that is implied in the limit of these fractions (which are values of the derived function) being always = 0. Consequently, each of $V_1$, $V_2$, &c. can be made as small a part as we please of $h$, and $V_1+V_2+\ldots\ldots+V_n$ as small a part as we please of $nh$, which is always $y$, or the sum preceding can be made as small a part as we please of a finite quantity, *i e.* diminishes without limit. Hence, $T(x+y)-T(y)$ a quantity independent of $h$, diminishes without limit with $h$, which is absurd; for, not containing $h$ at all, or any quantity whose value depends on $h$, it is the same whatever $h$ may be. Whence does this absurd conclusion arise? Not from the reasoning, but from the initial supposition, namely, $T(x+h)-T(x)$, &c. are quantities depending on $h$. If they depend on $h$ at all, all the preceding reasoning follows, and the inadmissible conclusion. The truth must be, then, that $T(x+h)-T(x)$ is not a function of $h$ at all. Neither then, does $\{T(x+h)-T(x)\}+T(x)$ contain $h$, for we have not added a function of $h$. That is, $T(x+h)$ is not a function of $h$; that is, $T(x)$ is not a function of $x$. For, if it were, $h$ would be found in $T(x+h)$. Therefore $T(x)$, supposed to be a function of $x$, and, therefore so expressed, turns out to be a quantity independent of $x$. Such a quantity is called a *constant* with respect to $x$. Hence the conclusion is, that if Q and R have the same derived function P, then Q can only differ from R by a constant quantity.

To verify the positive part of the conclusion, suppose $fx = Fx + C$, C not changing when $x$ changes. Then $f(x+h) = \mathrm{F}(x+h) + \mathrm{C}$, and

$$\frac{f(x+h) - fx}{h} = \frac{\mathrm{F}(x+h) - \mathrm{F}(x)}{h}$$

consequently the limits ($h$ diminishing) are the same, or $\mathrm{D}fx = \mathrm{DF}x$.

(129.) **Theorem.** If $\phi x$ be the derived function of $fx$, then the derived function of $f_{-1}x$, any inverse of $fx$ satisfying $ff_{-1}x = x$, is $\dfrac{1}{\phi(f_{-1}x)}$

$$\text{For} \quad \frac{f_{-1}(x+h) - f_{-1}x}{h} = \frac{f_{-1}(x+h) - f_{-1}x}{x+h-x} = \frac{f_{-1}(x+h) - f_{-1}x}{ff_{-1}(x+h) - ff_{-1}x}$$

$$= 1 \text{ divided by } \frac{ff_{-1}(x+h) - ff_{-1}x}{f_{-1}(x+h) - f_{-1}x}$$

Let $f_{-1}x = y$, $f_{-1}(x+h) = y+k$; then $k$ and $h$ diminish without limit together. Substitution gives

$$\frac{f_{-1}(x+h) - f_{-1}x}{h} = 1 \text{ divided by } \frac{f(y+k) - fy}{k}$$

Let $h$ diminish without limit, in which case $k$ does the same, and we have

$$\mathrm{D}f_{-1}x = 1 \text{ divided by } \mathrm{D}fy = \frac{1}{\phi y} = \frac{1}{\phi(f_{-1}x)}$$

For instance, let $fx = x^2$, $f_{-1}x = -\sqrt{x}$

$$\text{Then} \qquad \mathrm{D}fx = 2x = \varphi x \qquad \frac{1}{\phi x} = \frac{1}{2x}$$

$$\mathrm{D}(-\sqrt{x}) = \frac{1}{\phi(-\sqrt{x})} = \frac{1}{-2\sqrt{x}} = -\frac{1}{2\sqrt{x}}$$

Let $fx = \sin x$, then we make $f_{-1}x$ = an angle whose sine is $x$.

$$\mathrm{D}fx = \cos x = \varphi x \qquad \frac{1}{\phi x} = \frac{1}{\cos x}$$

$$\mathrm{D}f_{-1}x = \mathrm{D}\binom{\text{angle whose}}{\text{sine} = x} = \frac{1}{\phi(\text{ang. wh. sine is } x)} = \frac{1}{\cos(\text{angle, \&c.})}$$

$$= \frac{1}{\cos(\text{angle whose } \textit{cosine} \text{ is } (1-x^2)^{\frac{1}{2}})} = \frac{1}{(1-x^2)^{\frac{1}{2}}}$$

The ambiguity of the sign is a subject to be hereafter considered; when actual application takes place, it depends upon the following proposition.

(130.) $Dfx$ is positive for all values of $x$, at which, when $x$ increases or diminishes, $fx$ does the same; and negative when $fx$ diminishes by increase of $x$, or increases by diminution of $x$. We must suppose in this proposition, that the increments or decrements may be as small as we please; and the general algebraical meaning of increase and decrease is contemplated. Its converse is also true.

Firstly, let $f(x+h)$ be greater than $fx$, for any value of $h$, however small; that is (*Algebra*, p. 63.) let $f(x+h)-fx$ be positive. Then, $h$ being positive, we have $(f\overline{x+h}-fx) \div h$ is always positive, and its limit $Dfx$ must be positive. Conversely, let $Dfx$ be positive, then there are values of $h$ which are so small, that $f(x+h)-fx$ must be positive. For if we make

$$\frac{f(x+h)-fx}{h} = Dfx + H,$$

then $H$ must be an expression which diminishes without limit with $h$. The latter can then be taken so small, that $Dfx+H$ shall have the sign of $Dfx$, that is, shall be positive; that is, $h$ being positive, $h(Dfx+H)$ shall be positive, or $f(x+h)$ greater than $fx$.

The other cases of the direct and converse proposition can easily be established in the same way.

(131.) Let $x$ increase from $-a$ to $+a$, where $a$ may be as great as we please, so that we thus suppose a variation of $x$ between any limits of magnitude which it may be necessary to consider. We abbreviate all this into: Let $x$ increase from $-\infty$ to $+\infty$: then the preceding shews, that whenever ($x$ increasing) $Dfx$ changes from $+$ to $-$, the function ceases increasing, and begins decreasing, and *vice versâ*.

Apply this theorem to $D\sin x = \cos x$, and shew that the sign of the cosine corresponds to the current method of variation of the sine in the manner described in the theorem.

The following theorems may be established by the direct process in (126.). Symbols not containing $x$, are supposed independent of $x$.

If $\phi x = afx,\ D\phi x = aDfx$

If $\phi x = af_1x + bf_2x - cf_3x,\ D\phi x = aDf_1x + bDf_2x - cDf_3x$

$$Dx^m = \text{limit of} \frac{(x+h)^m - x^m}{h} = \text{limit}\left(mx^{m-1} + m\frac{m-1}{2}x^{m-2}h + \dots\right)$$

(*Algebra*, p. 217), or $Dx^m = mx^{m-1}$.

$$Dx^2 = 2x,\ Dx^3 = 3x^2,\ Dx^{\frac{1}{2}} = \tfrac{1}{2}x^{-\frac{1}{2}},\ Dx^{-1} = -x^{-2}, \&c.$$

$Dx =$ limit of $\frac{x+h-x}{h}$, or limit of 1. But 1 being invariable, as $h$ diminishes without limit, it remains the same, or $Dx = 1$. Similarly, $Dax = a$, $D(ax+b) = a$.

$$D(a_0 + a_1x + a_2x^2 + \ldots.) = a_1 + 2a_2x + 3a_3x^2 + \ldots.$$

This being universally true, we may write $a_2$ for $2a_2$, $a_3$ for $3a_3$, &c. which gives

$$D\left(a_0 + a_1x + a_2\frac{x^2}{2} + a_3\frac{x^3}{3} + \ldots\right) = a_1 + a_2x + a_3x^2 + \&c.$$

(132.) We have hardly thought it necessary to state that if two functions be always the same in value, their derived functions must be the same in value. But (124.) we must not take a limited form of one, and equate its derived function to that of a limited form of the other, without first ascertaining that our limited forms are really equals.

The derived function of a function of a function of $x$ is thus found.

Let us suppose $\phi(fx)$. Let $f(x+h) = fx + H$; then we have

$$D\phi fx = \text{limit of } \frac{\phi f(x+h) - \phi fx}{h} = \text{limit of } \frac{\phi(fx+H) - \phi fx}{H} \cdot \frac{H}{h}$$

$$= \text{limit of } \frac{\phi(fx+H) - \phi fx}{H} \times \text{limit of } \frac{f(x+h) - fx}{h}$$

Now $h$ and H diminish without limit together, and the first limit would be $D\phi x$, but that $fx$ is in the place of $x$. It is, therefore, what $D\phi x$ becomes when $fx$ is written for $x$. The second limit is $Dfx$; whence

$$D\phi fx = D\phi x \left(\begin{array}{c}\text{with } x \text{ afterwards} \\ \text{changed into } fx\end{array}\right) \times Dfx$$

$$\begin{array}{lrl} \text{Thus} & D\cos mx = & -\sin x\,(x \text{ changed into } mx) \times Dmx \\ & (131.) \quad = & -\sin mx \times m \\ & D\sin(x^5) = & \cos x \text{ (change } x \text{ into } x^5) \times Dx^5 \\ & = & \cos x^5 \times 5x^4\text{: and so on.} \end{array}$$

(133.) $D\log x$ is limit of $\frac{1}{h}\left(\log\overline{x+h} - \log x\right)$; or of $\frac{1}{h}\log\left(1+\frac{h}{x}\right)$;

or of $\frac{1}{h}\left(\frac{h}{x} - \frac{1}{2}\frac{h^2}{x^2} + \dots\right)$ (*Algebra*, p. 226); or of $\frac{1}{x} - \frac{1}{2}\frac{h}{x^2} + \dots$ that is,

$$\mathrm{D}\log x = \frac{1}{x} \qquad \mathrm{D}\log fx = \frac{1}{fx} \times \mathrm{D}fx$$

We shall now be able to proceed with the expression of the inverse trigonometrical functions, and with the extension of what has been already done with the direct functions.

# CHAPTER VII.

## CONTINUATION OF THE CONNEXION OF DIRECT AND INVERSE TRIGONOMETRICAL FUNCTIONS.

(134.) By $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, we mean in analogy with (122.) the angles which have $x$ for their sine, or cosine, or tangent, when we have both $\sin(\sin^{-1}x) = x$, and $\sin^{-1}(\sin x) = x$. That is, of all the possible angles contained in such a formula as $\theta \pm 2m\pi$, and having the same sine, there is one specific angle which is denoted by $\sin^{-1}x$, answering to the convertible inverse $f^{-1}x$. But all other angles contained in the same formula will be denoted by $\sin_{-1}x$, answering to the inconvertible inverses denoted generally by $f_{-1}x$.

Whatever a sine, or a tangent, may be in value, there is an angle lying between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, to which that sine or tangent belongs. Not so with the cosines, all of which are positive between those limits. But all cosines are found to belong to angles between 0 and $\pi$. Let the fundamental angles to which any given primary function is attached, be chosen between those limits. For instance, by $\sin^{-1}(-\frac{1}{2})$, is meant $-\frac{\pi}{6}$; but we have

$$\sin_{-1}(-\tfrac{1}{2}) = -\frac{\pi}{6} \pm 2m\pi, \text{ or } \pi - \left(-\frac{\pi}{6} \pm 2m\pi\right)$$

that is $$\frac{\pi}{6} \pm (2m+1)\pi$$

We find, in fact, the following general equations:

$$\sin_{-1}x = \sin^{-1}x \pm 2m\pi \quad \text{or} \quad \pm(2m+1)\pi - \sin^{-1}x$$
$$\cos_{-1}x = \cos^{-1}x \pm 2m\pi \quad \text{or} \quad \pm 2m\pi - \cos^{-1}x$$
$$\tan_{-1}x = \tan^{-1}x \pm 2m\pi \quad \text{or} \quad \tan^{-1}x \pm (2m+1)\pi$$

These are, in fact, but a combination of the propositions that there are in the first revolution two distinct angles, having the same sine, cosine, or tangent, and that any number of revolutions added or subtracted, does not alter the sine, cosine, or tangent.

(135.) We have shewn (129.) that

$$\mathrm{D}\sin^{-1}x = (1-x^2)^{-\frac{1}{2}},$$

but which sign we should take has not been determined. If we consider that while $x$ *increases* from $-1$ to $+1$, the angle *increases* from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, we see (130.) that the positive sign must be chosen, or that we have

$$\mathrm{D}\sin^{-1}x = \text{the positive value of } (1-x^2)^{-\frac{1}{2}}.$$

Expand this by the binomial theorem, which gives

$$\mathrm{D}\sin^{-1}x = 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \ldots\ldots$$

a series which (in common with all others of the kind derived from the binomial theorem, *Algebra*, p. 210) is convergent when $x^2 < 1$, or when $x$ lies between $+1$ and $-1$, that is, in every case we propose to apply it to. And by (131.) we have

$$\mathrm{D}\sin^{-1}x = \mathrm{D}\left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots\ldots\right)$$

But (128.) two expressions which have equal derived functions, can only differ by a quantity independent of $x$, whence we have

$$\sin^{-1}x = \mathrm{C} + x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \ldots\ldots$$

where C is as yet undetermined. We find it thus: since it is independent of $x$, whatever value it has for one value of $x$, it has the same for all. Let then $x = 0$ (read carefully *Algebra*, p. 189.) The only angle between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, which has 0 for sine, is 0, and the preceding then becomes $0 = \mathrm{C} + 0$, or $\mathrm{C} = 0$. That is,

$$\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots\ldots$$

Now, the only angle between 0 and $\pi$ which has $x$ for cosine, is $\frac{\pi}{2} - \sin^{-1}x$, so that we have

$$\cos^{-1}x = \frac{\pi}{2} - x - \frac{1}{2}\frac{x^3}{3} - \frac{1.3}{2.4}\frac{x^5}{5} - \ldots\ldots$$

which is the same result as we should obtain by going through the same process from the beginning, as we now proceed to do.

(136.) Returning to (129.), let $fx = \cos x$, then $\mathrm{D}fx$ or $\phi x = -\sin x$; therefore,

$$\mathrm{D}\cos^{-1}x = -\frac{1}{\sin(\cos^{-1}x)} = -\frac{1}{(1-x^2)^{\frac{1}{2}}}$$

where the sign is undetermined; but as $x$ diminishes from $+1$ to $-1$, while $\cos^{-1}x$ increases from 0 to $\pi$ (the limits (134.) between which it is contained), we must take a negative sign for $\mathrm{D}\cos^{-1}x$, or a positive sign for $(1-x^2)^{\frac{1}{2}}$. We have, therefore, by the same steps as before,

$$\mathrm{D}\cos^{-1}x = -\left(1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}.x^4 + ....\right)$$

or $\cos^{-1}x = \mathrm{C} - 1 - \frac{1}{2}x^2 - \frac{1.3}{2.4}x^4 - \&c.$

and C may be determined, as before, by making $x = 0$, which gives $\cos^{-1}0 = \mathrm{C}$. But between the limits of definition, $\frac{\pi}{2}$ is the only angle of which the cosine is 0, and we have, therefore,

$$\cos^{-1}x = \frac{\pi}{2} - x - \frac{1}{2}x^2 - \frac{1.3}{2.4}x^4 - \ .... \text{ as before.}$$

(137.) But we must now avoid all appearance of an unusual degree of caution without any particular reason shewn, by further examination of the proposition in (128.), namely, that if $\mathrm{D}\phi x = \mathrm{D}fx$, then $\phi x - fx$ must be a constant independent of $x$. That demonstration, conducted by reducing $\phi x - fx = \mathrm{T}x$, a real function of $x$, to an absurdity, supposed throughout that $\mathrm{DT}x$ presented, between $\mathrm{T}x$ and $\mathrm{T}(x+y)$, no peculiarities which would remove it out of ordinary rules, $x$ and $y$ being specific quantities: and also that $\mathrm{D}(\phi x - fx)$, or $\mathrm{D}\phi x - \mathrm{D}fx$, fulfilled the same condition. But there may happen cases in which $\phi(x+h) - \phi x$, or $f(x+h) - fx$, do not diminish without limit when $h$ diminishes without limit, or at least do not exhibit a form which will entitle us to draw that conclusion without further examination. If, for instance, $\phi x = \frac{1}{x}$, and $x = 0$, $x + h = h$, we can apply no conclusion of ordinary algebra to $\frac{1}{h} - \frac{1}{0}$, when $h$ diminishes without limit. We might investigate a large number of cases, but we shall confine ourselves to one, namely, that in which either of the derived functions becomes infinite. And first, having shewn that so long as there are derived functions $\mathrm{D}\phi x$ and $\mathrm{D}fx$, within the sense of the definition, we must arrive at $\mathrm{T}(x+y) = \mathrm{T}x$, or $\mathrm{T}x$ is independent of $x$; we may assume $\phi(x) = fx +$ (the same constant from $x = a$ to $x = b$) provided that between these limits $\mathrm{D}\phi x$ and $\mathrm{D}fx$ are finite; if, therefore, a change take place in the value of the constant, it must be at the point which our reasoning does not include. For example, let $\mathrm{D}fx$ be finite and intelligible from $x = a$ to $x = b$, and also from $x = b + k$ to $x = c$: but between $x = b$ and $x = b + k$ let there be a value which makes $\mathrm{D}fx = \propto$. We know, then, that if $\mathrm{D}\phi x = \mathrm{D}fx$,

from $x = a$ to $x = b$, $\varphi x = fx + \left\{\begin{array}{l}\text{(C, to be determined, but}\\ \text{of one value between}\\ \text{those values of } x).\end{array}\right.$

from $x = b + k$ to $x = c$, $\varphi x = fx +$ ($\mathrm{C}_1$ ditto ditto).

Are we at liberty to say that $\mathrm{C}_1$ must be C? Certainly not: for if we attempt to reason by the process in (128.), from $x = a$ to $x = c$, we include the case about which we can draw no conclusion, where $\mathrm{D}fx = \propto$, and where it is by no means evident that $\mathrm{V}_1 + \mathrm{V}_2 + \ldots\ldots + \mathrm{V}_n$ (in that article) will diminish without limit. We cannot shew, nor is it always true, that when $\mathrm{D}fx$ is infinite, a change takes place in the constant; but we have shewn the converse, namely, that if a change do take place in the constant, it must be when $\mathrm{D}fx$ undergoes some remarkable change of algebraical condition, either passing through infinity, becoming impossible, or the like. Let us suppose ourselves at a point at which the change of value takes place, and let it be when $x = t$. Then, when $x$ is $t - h$, the function in question is represented by $f(t-h) + \mathrm{C}$, but when $x = t + h$, by $f(t+h) + \mathrm{C}_1$.

Hence, calling $\phi x$ the general form of the function,

$$\varphi(t+h) - \varphi(t-h) = f(t+h) - f(t-h) + \mathrm{C}_1 - \mathrm{C}.$$

Now, in cases where $\mathrm{D}\phi x$ and $\mathrm{D}fx$ are finite for $x = t$, $\dfrac{\phi(t+h) - \phi(t-h)}{2h}$ and $\dfrac{f(t+h) - f(t-h)}{2h}$, will (127.), by diminishing $h$ without limit, give the derived functions of $\phi x$ and $fx$; and from the preceding it appears, that the difference of those derived functions will be the limit of $\mathrm{C}' - \mathrm{C}) \div h$, or infinite. But the difference of two finite quantities cannot be infinite; whence we may conclude, that $D\phi x$ and $Dfx$ are not both finite, and as they are always equal, both will increase without limit together.

(138.) The preceding results (135.) may be immediately made to confirm this conjectural reasoning, for it is hardly more: we see that from $x = -\frac{\pi}{2}$ to $x = +\frac{\pi}{2}$, we have

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \ldots.$$

which, when $x = 1$, gives a slowly converging series for $\frac{\pi}{2}$. But $x$ never increases beyond 1, therefore this series can never represent $2\pi + \sin^{-1} x$, which is the first value of $\sin_{-1} x$ greater than $\frac{\pi}{2}$, belonging to the supposition on which this series was deduced (135.), namely, that $x$ and $\sin^{-1} x$ increase together. In fact, if we suppose $x$ to vary from $-1$ to $+1$, $\sin^{-1} x$ *then increasing*, we shall find that

$$\begin{array}{rlccc}
\sin^{-1} x & \text{varies from} & -\frac{\pi}{2} & \text{to} & +\frac{\pi}{2} \\
2\pi + \sin^{-1} x & \text{varies from} & \frac{3\pi}{2} & \text{to} & \frac{5\pi}{2} \\
2m\pi + \sin^{-1} x & \text{varies from} & (4m-1)\frac{\pi}{2} & \text{to} & (4m+1)\frac{\pi}{2}
\end{array}$$

so that the general value of $\sin_{-1} x$ (when $x$ and $\sin^{-1} x$ increase together) is as follows. That value which lies between $4m - 1$ and $4m + 1$ right angles, is

$$2m\pi + x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \ldots.$$

for all values of $m$, positive and negative.

Similarly, the value of $\cos_{-1} x$, which lies between $4m$ and $4m + 2$ right angles (which includes the cases where $\cos^{-1} x$ decreases as $x$ increases, as in (136.)) is

$$\left(4m + 1\right)\frac{\pi}{2} - x - \frac{1}{2}\frac{x^3}{3} - \frac{1.3}{2.4}\frac{x^5}{5} - \ldots.$$

And at the points where the constant changes $x$ is $+1$, or $-1$, and $D\sin_{-1} x$, and $D\cos_{-1} x$ (129.) become infinite. In a note at the end of the work we shall collect the various cases, including the change in the form of the series itself, arising from the change of sign in the differential coefficients.

(139.) We have now to consider $\tan x$ and $\tan_{-1} x$. The developement of the former in a series of powers of $x$ is of little consequence as a result; but the methods by which it will here be obtained are important in other and more useful deductions.

THEOREM. A function of $x$, which is unaltered by changing the sign of $x$, or which satisfies the equation $\phi x = \phi(-x)$, cannot be expanded in a series of even and odd powers of $x$, but can only be expanded in even powers: and a function which is changed in sign only, and not in numerical value, by changing the sign of $x$, or which satisfies the equation $\phi(-x) = -\phi(x)$, can only be expanded in odd powers of $x$.

Let $\quad \phi x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ \cdots\cdot$

Then $\phi(-x) = a_0 - a_1 x + a_2 x^2 - a_3 x^3 + a_4 x^4 - \ \cdots\cdot$

If $\phi(x) = \phi(-x)$, addition and division by 2 gives

$$\phi x = a_0 + a_2 x^2 + a_4 x^4 + \ \cdots\cdot$$

If $\phi(-x) = -\phi x$, subtraction and division by 2 gives

$$\phi x = a_1 x + a_3 x^3 + a_5 x^5 + \ \cdots\cdot$$

In the first case, then, $a_1 x + a_3 x^3 + \ \cdot\cdot = 0$ or (*Algebra*, p. 188)

$$a_1 = 0 \quad a_3 = 0 \text{ \&c.}$$

In the second case, $\quad a_0 + a_2 x^2 + \ \cdot\cdot = 0$ or (*Algebra*, p. 188)

$$a_0 = 0 \quad a_2 = 0 \text{ \&c.}$$

(140.) The derived function of $\tan x$ is $\frac{1}{\cos^2 x}$, or $1 + \tan^2 x$; for (56.)

$$\frac{\tan(x+h) - \tan x}{h} = \frac{\sin(x+h-x)}{h.\cos(x+h)\cos x} = \frac{1}{\cos(x+h)\cos x} \cdot \frac{\sin h}{h}$$

the limit of which gives $\mathrm{D}\tan x = \frac{1}{\cos^2 x} = 1 + \tan^2 x$.

Again, $\tan x$, when the sign of $x$ is changed, becomes $-\tan x$, or $\tan(-x) = -\tan x$; whence (139.) the series for $\tan x$ must consist of odd powers only. We have, therefore, to investigate what series of odd powers, P, will give $\mathrm{DP} = 1 + \mathrm{P}^2$.

Let $\quad \mathrm{P} = a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 + a_9 x^9 + \ \cdots\cdot$

Then, $\mathrm{DP} = a_1 + 3a_3 x^2 + 5a_5 x^4 + 7a_7 x^6 + 9a_9 x^8 + \ \cdots\cdot$

$$\begin{aligned} P^2 = a_1^2x^2 + 2a_1a_3x^4 + 2a_1a_5x^6 + 2a_1a_7x^8 + & \dots \\ + a_3^2x^6 + 2a_3a_5x^8 + & \dots \\ + & \dots \end{aligned}$$

Make DP equivalent with $1 + P^2$, which gives

$$a_1 = 1,\ 3a_3 = a_1^2 \text{ or } a_3 = \frac{1}{3},\quad 5a_5 = 2a_1a_3 \text{ or } a_5 = \frac{2}{15},$$

$$7a_7 = 2a_1a_5 + a_3^2 = \frac{4}{15} + \frac{1}{9} = \frac{17}{3.15} \text{ or } a_7 = \frac{17}{3.7.15},$$

$$9a_9 = 2a_1a_7 + 2a_3a_5 = \frac{34}{3.7.15} + \frac{4}{3.15} = \frac{62}{3.7.15},\ a_9 = \frac{62}{3.7.9.15}$$

whence, $x + \frac{1}{3}x^3 + \frac{2}{15}x^6 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \dots$

is the only series of odd powers which has a property such as the tangent must have; namely, to be a function of $x$, whose derived function is $1 +$ the square of the function. But $\tan x$, if developable at all in powers of $x$, must be developable in *odd* powers only (139.); consequently, the preceding series must be the tangent of $x$, if there be any series whatever of whole powers of $x$ which is $= \tan x$. But, by common division, we may see that $\sin x$, or $x - \frac{x^3}{2.3} + \dots$ divided by $\cos x$, or $1 - \frac{x^2}{2} + \dots$ does give an equivalent series of powers of $x$; consequently, we must have

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \dots$$

The actual division of the sine by the cosine will verify this, as follows:

$$\begin{array}{l} 1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \dots \Big) x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \dots \Big( x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots \\ \qquad\qquad\qquad\quad \underline{x - \frac{x^3}{2} + \frac{x^5}{2.3.4} - \dots} \\ \qquad\qquad\qquad\qquad\quad \frac{1}{3}x^3 - \frac{4}{2.3.4.5}x^5 + \dots \\ \qquad\qquad\qquad\qquad\quad \underline{\frac{1}{3}x^3 - \frac{1}{2.3}x^5 + \dots} \\ \qquad\qquad\qquad\qquad\qquad\qquad \frac{4}{2.3.5}x^5 + \dots \end{array}$$

The preceding series must cease to be convergent when $x = \frac{\pi}{2}$, if not before; and must, therefore, be regarded (*Algebra*, chap. ix.) as an *algebraical* equivalent of the tangent.

(141.) We shall now proceed to the determination of $\tan_{-1} x$. Let us first take that value which lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, (134.) which we have agreed to call $\tan^{-1} x$.

From (101.) $$\tan x = \frac{1}{\sqrt{-1}} \frac{\varepsilon^{x\sqrt{-1}} - \varepsilon^{-x\sqrt{-1}}}{\varepsilon^{x\sqrt{-1}} + \varepsilon^{-x\sqrt{-1}}} = \frac{1}{\sqrt{-1}} \frac{\varepsilon^{2x\sqrt{-1}} - 1}{\varepsilon^{2x\sqrt{-1}} + 1}$$

the latter fraction being made by multiplying numerator and denominator of the preceding by $\varepsilon^{x\sqrt{-1}}$. From this we find

$$\varepsilon^{2x\sqrt{-1}} = \frac{1 + \sqrt{-1}\,.\tan x}{1 - \sqrt{-1}\,\tan x}$$ and (*Algebra*, p. 226, formula 3.)

$$\log \varepsilon^{2x\sqrt{-1}} = 2\left(\sqrt{-1}\,.\tan x + \tfrac{1}{3}(\sqrt{-1}\tan x)^3 + \tfrac{1}{5}(\sqrt{-1}\tan x)^5 + \ldots\ldots\right)$$
$$= 2\sqrt{-1}\,(\tan x - \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x - \&c.)$$

Before proceeding further, annex $2\pi m\sqrt{-1}$ to the second side (102.), because (124.) we have no right to conclude that any one logarithm of the first side is equal to any one logarithm of the other. And, to make the question clear of all difficulties, except one, take a specific value for the tangent of $x$, say $\cdot 1$. We want to determine the angle or angles which have $\cdot 1$ for their tangent. Calculate

$$\cdot 1 - \frac{\cdot 001}{3} + \frac{\cdot 00001}{5} - \&\text{c. and call it } v.$$

Then, taking the general logarithms of both sides, we have

$$2x\sqrt{-1} + 2\pi n\sqrt{-1} = 2v\sqrt{-1} + 2\pi m\sqrt{-1}$$

or $\quad x + n\pi = v + m\pi, \quad$ and $\quad x = v + (m-n)\pi,$

which is the same as $\quad x = v + p\pi,$

where $p$ is any whole number, positive or negative. By proceeding similarly with the general series, we obtain

$$x = (\tan x - \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x - \&\text{c.}) + p\pi,$$

a result of a true form; for (44.) whatever angle $v$ has a given tangent, $v + p\pi$ has the same. The only question is, with a given tangent, to which of the angles does the series itself belong? To investigate this, we shall produce the series itself by aid of the last chapter. Firstly, we must investigate $\mathrm{D}\tan_{-1}x$ (129.)

Here $\quad fx = \tan x,\ \mathrm{D}fx = \varphi x = 1 + \tan^2 x,$

$$\mathrm{D}f_{-1}x = \frac{1}{\phi(f_{-1}x)} = \frac{1}{1+(\tan.\tan_{-1}x)^2} = \frac{1}{1+x^2}$$

or $\quad \mathrm{D}\tan_{-1}x = \dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots\ldots$

(*Algebra*, p. 16), from which, by reasoning similar to that in (135.)

$$\tan_{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots. + \mathrm{C}.$$

Now, if we look at (128.), we see that the only supposition on which the reasoning there given can fail, is where $x$ itself is infinite; for $\dfrac{1}{1+x^2}$ never becomes infinite for possible values of $x$. Consequently, between $\tan_{-1}x = -\dfrac{\pi}{2}$, and $\tan_{-1}x = +\dfrac{\pi}{2}$ there is no change of value of the constant. Let $\mathrm{C}_1$ be the value of the constant which belongs to the interval between $-\dfrac{\pi}{2}$ and $+\dfrac{\pi}{2}$ (in which $\tan_{-1}x$ is $\tan_{-1}x$); then

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots + \mathrm{C}_1;$$

but when $x = 0$, $\tan^{-1}x = 0$, and therefore, as in (135.), $\mathrm{C}_1 = 0$, or we have

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$$

for $x$ write $\tan x$, and (134.) we have

$$x = \tan x - \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x - \&c.$$

that is, the value of the series is the angle which has the tangent specified in it, and which lies between $-\dfrac{\pi}{2}$ and $+\dfrac{\pi}{2}$.

This series is convergent only when $\tan x$ is less than 1. But we may immediately produce a series which shall always be convergent, as follows. From (135.)

$$\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \ldots\ldots$$

for $x$ write $\sin x$, which gives

$$x = \sin x + \frac{1}{2}\,\frac{1}{3}\,\sin^3 x + \frac{1.3}{2.4}\,\frac{1}{5}\,\sin^5 x + \dots\dots$$

for $\sin x$ write $\tan x \div \sqrt{(1 + \tan^2 x)}$, and then write $\tan^{-1} x$ for $x$, which gives

$$\tan^{-1} x = \frac{1}{\sqrt{1+x^2}}\left\{1 + \frac{1}{2}\,\frac{1}{3}\,\frac{1}{1+x^2} + \frac{1.3}{2.4}\cdot\frac{1}{5}\,\frac{1}{(1+x^2)^2} + \dots\right\}$$

which will be found (*Algebra*, p. 182) to be always convergent.

(142.) As an exercise, deduce from $x = \tan x - \frac{1}{3}\tan^3 x +$ &c. by taking derived functions of both sides, the following series:

$$\left.\begin{aligned}\cos^2 x &= 1 - \tan^2 x + \tan^4 x - \dots\dots\dots\\ \sin^2 x &= \tan^2 x - \tan^4 x + \tan^6 x - \dots\dots\end{aligned}\right\}\text{and verify them otherwise.}$$

(143.) A series of periodic quantities, such as

$$\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta + \dots\dots \textit{ ad inf.}$$

can neither be called convergent nor divergent. If for $\theta$ we take any value, these sines will be a succession of positive and negative quantities, in parcels, no one of which exceeds unity. If, for example, we had $\theta = \frac{\pi}{4}$, the series would be

$$\left(\tfrac{1}{2}\sqrt{2} + 1 + \tfrac{1}{2}\sqrt{2} + 0 - \tfrac{1}{2}\sqrt{2} - 1 - \tfrac{1}{2}\sqrt{2} - 0\right)$$

+ a second parcel of the same form;

+ a third; + ........ *ad inf.*

to which we may give the form

$$(1 + \sqrt{2})\,\{1 - 1 + 1 - 1 + \dots\dots\dots \textit{ ad inf.}\}$$

The meaning of this series has been already discussed in *Algebra*, p. 197. The question is, are we now to say that the preceding series is *one-half* of $1 + \sqrt{2}$. In the chapter cited, we found the series $1 - 1 + 1 - 1 + \dots\dots$ to be a (then) unintelligible limiting form of $1 - x + x^2 - x^3 + \dots\dots$ which was always arithmetically equal to $\frac{1}{1+x}$, as long as $x$ was less than 1; whence we assumed $\frac{1}{2} = 1 - 1 + 1 - \dots\dots$, and found that the one side was an algebraical equivalent of the other. Let us then examine the series

$$x\sin\theta + x^2\sin 2\theta + x^3\sin 3\theta + \dots\dots \quad (x < 1),$$

which we may now call convergent, as its terms are severally less than those of the series $x + x^2 + x^3 + \ldots.$, which *is* convergent. We may sum this series in various ways, of which we shall choose two.

1. $\sin(n+1)\theta + \sin(n-1)\theta = 2\sin n\theta . \cos\theta$ gives

$$n = 1,\ x\sin 2\theta + 0 = 2x\sin\theta . \cos\theta$$

$$n = 2,\ x^2\sin 3\theta + x^2\sin\theta = 2x^2\sin 2\theta . \cos\theta$$

$$n = 3,\ x^3\sin 4\theta + x^3\sin 2\theta = 2x^3\sin 3\theta . \cos\theta \ \text{\&c.}$$

If, then, we call the sum of the whole series S, and sum these equations *ad infinitum*, the sum of the first column, $x\sin 2\theta + x^2 \sin 3\theta + \ldots.$ is $(S - x\sin\theta) \div x$; that of the second column is $xS$; that of the third column is $2S\cos\theta$. Consequently,

$$\frac{S - x\sin\theta}{x} + xS = 2S\cos\theta, \quad \text{or} \quad S = \frac{x\sin\theta}{x^2 - 2x\cos\theta + 1}.$$

2. Let the sines be replaced by their exponential values, which gives

$$(\varepsilon^{\theta\sqrt{-1}} = y) \quad \frac{1}{2\sqrt{-1}}\left\{x\left(y - \frac{1}{y}\right) + x^2\left(y^2 - \frac{1}{y^2}\right) + \ldots.\right\}$$

$$= \frac{1}{2\sqrt{-1}}\left(xy + x^2y^2 + \ldots.\right) - \frac{1}{2\sqrt{-1}}\left(\frac{x}{y} + \frac{x^2}{y^2} + \ldots.\right)$$

$$= \frac{1}{2\sqrt{-1}}\,\frac{xy}{1 - xy} - \frac{1}{2\sqrt{-1}}\,\frac{xy^{-1}}{1 - xy^{-1}} = \frac{1}{2\sqrt{-1}}\,\frac{xy - xy^{-1}}{1 - (y + y^{-1})x + x^2}$$

$$= \frac{x}{1 - (y + y^{-1})x + x^2}\,\frac{y - y^{-1}}{2\sqrt{-1}} = \frac{x\sin\theta}{1 - 2\cos\theta . x + x^2}$$

the same as before. Now, if in this result we make $x = 1$, we find it become

$$\frac{\sin\theta}{2(1 - \cos\theta)} \quad \text{or} \quad \frac{2\sin\frac{1}{2}\theta . \cos\frac{1}{2}\theta}{4\sin^2\frac{\theta}{2}} \quad \text{or} \quad \tfrac{1}{2}\cot\frac{\theta}{2}$$

that is, $\frac{1}{2}\cot\frac{\theta}{2} = \sin\theta + \sin 2\theta + \sin 3\theta + \ldots.$

on which we must remark, that arithmetical equality has ceased, and that the sign = can only mean (*Algebra*, chap. ix.) that one side may be substituted for the other without producing discordances in any arithmetical consequence of the substitution. For instance, we shall consider the following result, which those who have any idea of arithmetical identity between the two will be surprised at. The smaller we make $\theta$, the greater is the value of the series. On which remark, that this is not a series of terms with fixed signs, but one in which the signs are positive and negative in parcels, the number of terms in each parcel depending upon the value of $\theta$. First, let $\theta$ be diminished until the first three angles lie in the first two right angles the signs then are,

$$+ + + - - \ldots\ldots$$

let further diminution take place until six angles are less than $\pi$; the signs are then,

$$+ + + + + + - - \ldots\ldots$$

and so on. Now, one very simple case of the series is that in whi $\theta$ is a measure of $\pi$, say $n\theta = \pi$. We have then $\sin(n\theta + \theta) = -\sin\theta$, $\sin(n\theta + 2\theta) = -\sin 2\theta$, &c.; so that, if we put $a$ for $\sin\theta + \sin 2\theta + \ldots + \sin n\theta$, we have $a - a + a - a + \ldots$ or $\frac{1}{2}a$ for the series. It remains then, only to find an expression for $a$; now we have

$$a = \frac{1}{2\sqrt{-1}}\left(y - y^{-1} + y^2 - y^{-2} + \cdots + y^n - y^{-n}\right)$$

$$= \frac{1}{2\sqrt{-1}}\left\{\frac{y - y^{n+1}}{1 - y} - \frac{y^{-1} - y^{-n-1}}{1 - y^{-1}}\right\}$$

$$= \frac{1}{2\sqrt{-1}}\,\frac{y - y^{-1} + y^n - y^{-} - (y^{n+1} - y^{-n-1})}{2 - (y + y^{-1})}$$

$$= \frac{\sin\theta + \sin n\theta - \sin(n+1)\theta}{2(1 - \cos\theta)}$$

which, if $n\theta = \pi$, $\sin n\theta = 0$, $\sin(n+1)\theta = -\sin\theta$, gives $a = \frac{2\sin\theta}{2(1 - \cos\theta)}$; and $\frac{a}{2}$ is the sum of the series already found.

(144.) If $\theta = \frac{m}{n}\pi$, $m$ and $n$ being whole numbers, and if $\mathrm{F}\theta$ be a primary function of $\theta$, or any function whatsoever of primary functions which is really periodic, the series $\mathrm{F}\theta + \mathrm{F}2\theta + \ldots$ can always be reduced to the form $a - a + a - a + \ldots$; and the whole question of assigning an algebraical equivalent to the series, in *finite* terms, is, therefore, made to depend upon the considerations developed in the ninth chapter of my *Algebra*. And, when $\theta$ is not commensurable with $\pi$, we know that $\theta$ may be found to lie between $\frac{m}{n}\pi$ and $\frac{m+1}{n}\pi$ where $n$ is greater than any number named: so that the question in any other case is resolved into the general question discussed in the treatise on Number and Magnitude.

(145.) When both the series and its algebraical invelopement* are periodic in value, there is generally no difficulty with regard to the constants which may enter. But it may happen that one side increases without limit while the other side is periodic, in which case a discontinuous constant must enter.

For instance, let the series be

$$\sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta + \ldots.$$

which differs from that in (143.) by having terms which decrease without limit; but which are positive and negative in parcels determined by the value of $\theta$. This series is,

$$\frac{1}{2\sqrt{-1}}\left\{y - y^{-1} + \frac{1}{2}\left(y^2 - y^{-2}\right) + \frac{1}{3}\left(y^3 - y^{-3}\right) + \ldots.\right\}$$

Now, (*Algebra*, p. 226) $y + \frac{1}{2}y^2 + \frac{1}{3}y^3 + \ldots = -\log(1-y)$

$$y^{-1} + \frac{1}{2}y^{-2} + \frac{1}{3}y^{-3} + \ldots = -\log\left(1 - \frac{1}{y}\right)$$

whence the series $= \dfrac{1}{2\sqrt{-1}}\left\{\log\left(1-\frac{1}{y}\right) - \log(1-y)\right\}$

$$= \frac{1}{2\sqrt{-1}}\log\frac{y-1}{y(1-y)} = \frac{1}{2\sqrt{-1}}\log\left(-\frac{1}{y}\right) = \frac{1}{2\sqrt{-1}}\log \varepsilon^{-(\theta+\pi)\sqrt{-1}}$$

Because $\frac{1}{y} = \varepsilon^{-\theta\sqrt{-1}}$ $-\frac{1}{y} = \varepsilon^{-(\theta+\pi)\sqrt{-1}}$; and, as in (102.), the general logarithm gives

$$\sin\theta + \frac{1}{2}\sin 2\theta + \ldots = \frac{2\pi n\sqrt{-1} - (\theta+\pi)\sqrt{-1}}{2\sqrt{-1}} = \pi n - \frac{\theta+\pi}{2}$$

It remains to determine the value of $n$, corresponding to the condition that $\theta$ lies between certain limits, which limits are also to be found. Let $\theta = \frac{\pi}{2}$; the series becomes $1 - \frac{1}{3} + \frac{1}{5} - \ldots.$ or (141.) $\tan^{-1}1$, that is $\frac{\pi}{4}$, whence we find

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* I use this word as opposed to *developement;* thus $\frac{1}{2}$ is the invelopement of $1-1+1-1+\ldots\ldots$

$$\pi n - \frac{3\pi}{4} = \frac{\pi}{4}, \text{ or } n = 1 \text{ when } \theta = \frac{\pi}{2}$$

Similarly, $$\pi n - \frac{\pi}{4} = -\frac{\pi}{4} \text{ or } n = 0 \text{ when } \theta = -\frac{\pi}{2}$$

Now, the addition of $2\pi$ to the first side of the equation creates no difference; neither must it do so on the second side. To satisfy this condition, consistently with those just deduced, and also with another which appears immediately, namely, that a change of sign in $\theta$ changes the sign of the first side, and, therefore, of the second, let $n_1$ be the value of $n$ between $\theta = 0$ and $\theta = \frac{\pi}{2}$, and $n_2$ between $\theta = 0$ and $\theta = -\frac{\pi}{2}$. We have then

$$\pi n_1 - \frac{\theta + \pi}{2} = -\left(\pi n_2 - \frac{-\theta + \pi}{2}\right) \quad \text{or} \quad n_1 = -n_2 + 1$$

the only value of which is $n_2 = 0$, $n_1 = 1$, since the only change consistent with periodicity is the addition of $\pi$, whenever $\theta$ has received an increase of $2\pi$: and since $n$ is 1 when $\theta = \frac{\pi}{2}$, and 0 when $\theta = -\frac{\pi}{2}$, and the change takes place when $\theta$ changes from negative to positive, it must be as just stated. Such an expression as $a - \frac{\theta}{2}$ can only thus be made identical with a periodic series; for, when $\theta$ becomes $\theta + 2\pi$, then the preceding becomes $a - \pi - \frac{\theta}{2}$, in which, if the same succession of values is to recur while the angle changes from $\theta + 2\pi$ to $\theta + 4\pi$, $a$ must be increased by $\pi$, so that $a - \pi$ may be then what $a$ was. The result is that

from $\theta = -2\pi$ to $\theta = 0$ the series is $-\frac{\theta+\pi}{2}$ lying between $\frac{\pi}{2}$ and $-\frac{\pi}{2}$

$\theta = 0$ to $\theta = 2\pi$ …… $\pi - \frac{\theta+\pi}{2}$ …… $\frac{\pi}{2}$ .. $-\frac{\pi}{2}$

$\theta = 2\pi$ to $\theta = 4\pi$ …. $2\pi - \frac{\theta+\pi}{2}$ …… $\frac{\pi}{2}$ .. $-\frac{\pi}{2}$

and so on.

(146.) Let us consider the series

$$a \sin\theta + \frac{a^2}{2}\sin 2\theta + \frac{a^3}{3}\sin 3\theta + \dots$$

which is reduced to the preceding by $a = 1$. Substitute for $\sin\theta$, as in (145.), and the preceding becomes

$$\frac{1}{2\sqrt{-1}}\left\{\log\left(1-\frac{a}{y}\right)-\log(1-ay)\right\} \quad \text{or} \quad \frac{1}{2\sqrt{-1}}\log\frac{1-ay^{-1}}{(1-ay)}$$

But $\quad y = \cos\theta + \sqrt{-1}\sin\theta \qquad \frac{1}{y} = \cos\theta - \sqrt{-1}\sin\theta$

whence the series is $\quad \dfrac{1}{2\sqrt{-1}}\log\dfrac{1-a\cos\theta+a\sin\theta\sqrt{-1}}{1-a\cos\theta-a\sin\theta\sqrt{-1}}$

Let $1-a\cos\theta = r\cos\phi$, $a\sin\theta = r\sin\phi$, whence $\phi = \tan_{-1}\left(\dfrac{a\sin\theta}{1-a\cos\theta}\right)$. The value of the series becomes

$$\frac{1}{2\sqrt{-1}}\log\frac{\cos\phi+\sin\phi\sqrt{-1}}{\cos\phi-\sin\phi\sqrt{-1}} = \frac{1}{2\sqrt{-1}}\log\frac{\varepsilon^{\phi\sqrt{-1}}}{\varepsilon^{-\phi\sqrt{-1}}}$$

$$= \frac{1}{2\sqrt{-1}}\log\varepsilon^{2\phi\sqrt{-1}} = \frac{2\phi\sqrt{-1}+2n\pi\sqrt{-1}}{2\sqrt{-1}} = \phi + n\pi$$

$$a\sin\theta + \frac{a^2}{2}\sin 2\theta + \dots\dots = n\pi + \tan^{-1}\left(\frac{a\sin\theta}{1-a\cos\theta}\right)$$

where $\tan_{-1}$ is changed into $\tan^{-1}$, as the difference between the two is expressed in the remaining term. In the form here given, $n$ will have only one value; for $\tan^{-1}x$ is not a continually increasing term, being limited to that angle which lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$. And, when $a$ is very small, the preceding series is very small, which cannot be true of the second side, unless $n = 0$. Consequently, we have

$$\tan^{-1}\left(\frac{a\sin\theta}{1-a\cos\theta}\right) = a\sin\theta + \frac{a^2}{2}\sin 2\theta + \dots\dots$$

This brings us to consider the difference between $\tan^{-1}\tan\theta$ and $\theta$, which are equals only when $\theta$ lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$: but, if $\theta$ be greater than $\frac{\pi}{2}$, it is not $\tan^{-1}\tan\theta$ which is $=\theta$, but one of the values of $\tan_{-1}\tan\theta$. Consequently, as $\theta$ increases in successive revolutions, the expression $\tan^{-1}\tan\theta$ is periodic, varying continually between $\frac{\pi}{2}$ and $-\frac{\pi}{2}$. And similar considerations may be applied to $\sin^{-1}\sin\theta$, which has the same limits, and to $\cos^{-1}\cos\theta$ which has the limits 0 and $\pi$. We shall immediately see the use of this when we proceed to the series of last article, considered as a particular case of the present one. Let $a = 1$, then we have

$$\frac{a\sin\theta}{1-a\cos\theta} = \frac{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$

or $\quad \tan^{-1}\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right) = \sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta + \ldots.$

if we write the first side $\frac{\pi}{2}-\frac{\theta}{2}$, we are in error, for the two are not the same, except when $\frac{\pi}{2}-\frac{\theta}{2}$ lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, or when $\theta$ lies between 0 and $2\pi$. Such a transition would require a continual correction, amounting, in fact, to the process of the last article: but $\tan^{-1}\tan x$ is, by definition, the angle which lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, and has $x$ for its tangent.

The student should now prove the following theorems, illustrative of the limitation of $F^{-1}$, as distinguished from $F_{-1}$.

$$\sin^{-1}\cos x = \frac{\pi}{2} - x \text{ from } x = 0 \text{ to } x = \pi$$

$$\cos^{-1}\sin x = \frac{\pi}{2} - x \text{ from } x = -\frac{\pi}{2} \text{ to } x = +\frac{\pi}{2}$$

$$\cot^{-1}\tan x = \frac{\pi}{2} - x \text{ from } x = 0 \text{ to } x = +\pi$$

(147.) We now proceed to some adaptations of the theorems in chap. ii. to the notation of inverse functions.

$$\sin(\sin^{-1}x + \sin^{-1}y) = \sin\sin^{-1}x.\cos\sin^{-1}y + \cos\sin^{-1}x.\sin\sin^{-1}y$$
$$= x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

$$\cos(\sin^{-1}x + \sin^{-1}y) = \sqrt{1-x^2}.\sqrt{1-y^2} - xy$$

$$\sin(\sin^{-1}x + \cos^{-1}y) = xy + \sqrt{1-x^2}\ \sqrt{1-y^2}$$

$$\sin(2\sin^{-1}x) = 2x\sqrt{1-x^2} \qquad \cos(2\sin^{-1}x) = 1-2x^2$$

$$\sin(2\cos^{-1}x) = 2x\sqrt{1-x^2} \qquad \cos(2\cos^{-1}x) = 2x^2-1$$

Is it true that $\sin(2\sin^{-1}x) = \sin(2\cos^{-1}x)$? if not, why? Investigate the consequences of the ambiguity of sign implied in $\sqrt{\phantom{xx}}$, and also ascertain within what limits the following is true, and when and how it must be corrected.

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$

Next, $\tan(\tan^{-1}x + \tan^{-1}y) = \dfrac{x+y}{1-xy}$, $\tan(2\tan^{-1}x) = \dfrac{2x}{1-x^2}$

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

This latter equation is true whenever the sum of $\tan^{-1}x$ and $\tan^{-1}y$ lies between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$: but if it do not, say, for instance, it lies between $\dfrac{\pi}{2}$ and $\pi$, then either $\pi$ must be subtracted from the first side, or $\tan_{-1}$ must be used on the second; and so on. In the same manner deduce (with similar limitation),

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

(148.) Let $\dfrac{x+y}{1-xy} = 1$ or $y = \dfrac{1-x}{1+x}$ which gives

$$\tan^{-1}x + \tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{\pi}{4}$$

Let $x = \dfrac{1}{2}$, $y = \dfrac{1}{3}$, and then $\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3} = \dfrac{\pi}{4}$, which gives an easy method of computing the value of $\pi$. For (141.)

$$\tan^{-1}\frac{1}{2} = \frac{1}{2} - \frac{1}{3}\,\frac{1}{2^3} + \frac{1}{5}\,\frac{1}{2^5} - \frac{1}{7}\,\frac{1}{2^7} + \dots\dots$$

$$\tan^{-1}\frac{1}{3} = \frac{1}{3} - \frac{1}{3}\,\frac{1}{3^3} + \frac{1}{5}\,\frac{1}{3^5} - \frac{1}{7}\,\frac{1}{3^7} + \dots\dots$$

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{3} - \frac{1}{3}\left(\frac{1}{2^3} + \frac{1}{3^3}\right) + \frac{1}{5}\left(\frac{1}{2^5} + \frac{1}{3^5}\right) - \text{\&c.}$$

Write $p$ and $q$ for $\dfrac{1}{2}$ and $\dfrac{1}{3}$ (divide by 4 and 9 at every step).

$p = \cdot 50000000000$
$p^3 = \cdot 12500000000$
$p^5 = \cdot 03125000000$
$p^7 = \cdot 00781250000$
$p^9 = \cdot 00195312500$
$p^{11} = \cdot 00048828125$
$p^{13} = \cdot 00012207031$
$p^{15} = \cdot 00003051758$
$p^{17} = \cdot 00000762940$
$p^{19} = \cdot 00000190735$
$p^{21} = \cdot 00000047684$
$p^{23} = \cdot 00000011921$
$p^{25} = \cdot 00000002980$
$p^{27} = \cdot 00000000745$
$p^{29} = \cdot 00000000186$
$p^{31} = \cdot 00000000047$
$p^{33} = \cdot 00000000012$
$p^{35} = \cdot 00000000003$

$q = \cdot 33333333333$
$q^3 = \cdot 03703703704$
$q^5 = \cdot 00411522634$
$q^7 = \cdot 00045724737$
$q^9 = \cdot 00005080526$
$q^{11} = \cdot 00000564503$
$q^{13} = \cdot 00000062723$
$q^{15} = \cdot 00000006969$
$q^{17} = \cdot 00000000774$
$q^{19} = \cdot 00000000086$
$q^{21} = \cdot 00000000009$

Now let $(p^n + q^n) \div n$ be denoted by $r_n$.

$$\begin{array}{rcl@{\qquad}rcl}
r_1 & = & \cdot 83333333333 & r_3 & = & \cdot 05401234568 \\
r_5 & = & \cdot 00707304527 & r_7 & = & \cdot 00118139248 \\
r_9 & = & \cdot 00022265892 & r_{11} & = & \cdot 00004490239 \\
r_{13} & = & \cdot 00000943827 & r_{15} & = & \cdot 00000203915 \\
r_{17} & = & \cdot 00000044924 & r_{19} & = & \cdot 00000010043 \\
r_{21} & = & \cdot 00000002271 & r_{23} & = & \cdot 00000000518 \\
r_{25} & = & \cdot 00000000119 & r_{27} & = & \cdot 00000000028 \\
r_{29} & = & \cdot 00000000006 & r_{31} & = & \cdot 00000000002 \\
\cline{3-3}\cline{6-6}
 & & \cdot 84063894899 & & & \cdot 05524078561 \\
 & & & & & \cdot 84063894899 \\
\cline{6-6}
 & & & \frac{1}{4}\pi & = & \cdot 78539816338 \\
 & & & & & 4 \\
\cline{6-6}
 & & & \pi & = & 3\cdot 14159265352
\end{array}$$

which is correct, with the exception of the last place.

(149.) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$, &c. Form all similar equations, and explain under what limitations they are true.

It must be observed that every such equation becomes true (134.) when $\sin_{-1}$, &c. are substituted for $\sin^{-1}$, &c. But it does not follow that the same value of $m$ in the general equation

$$\sin_{-1}x = 2m\pi + \sin^{-1}x = (2m+1)\pi - \sin^{-1}x$$

must be applied on both sides of the equation.

(150.) The angle $\sin^{-1}x$ is made, *by convention*, a *periodic* angle. It changes through $0, \frac{\pi}{2}, 0, -\frac{\pi}{2}, 0$, &c. while $x$ changes through $0, 1, 0, -1, 0$, &c.

If $x$ be greater than 1, the angle $\sin^{-1}x$ becomes impossible. If in the expressions for $\sin x$ and $\cos x$, we write $x\sqrt{-1}$ for $x$, we have

$$\sin\left(x\sqrt{-1}\right) = \frac{\varepsilon^{-x} - \varepsilon^{x}}{2\sqrt{-1}} \quad \text{or} \quad \frac{\varepsilon^{x} - \varepsilon^{-x}}{2} = \frac{\sin\left(x\sqrt{-1}\right)}{\sqrt{-1}}$$

$$\cos\left(x\sqrt{-1}\right) = \frac{\varepsilon^{-x} + \varepsilon^{x}}{2} \quad \text{or} \quad \frac{\varepsilon^{x} + \varepsilon^{-x}}{2} = \cos\left(x\sqrt{-1}\right).$$

The right hand sides of the equations are no longer periodic; and in the same way as all functions of sines, cosines, &c. may be expressed by exponential functions of $x$ and $\sqrt{-1}$, so all exponential functions may be expressed by forms of sines and cosines of $x\sqrt{-1}$.

(151.) We may thus always give to periodic series their correct periodic values, either in terms of the primary functions, sines, &c. which are periodic, or by the introduction of *periodic angles*, $\sin^{-1}\sin\theta$, $\cos^{-1}\cos\theta$, &c. Having illustrated this point, on which freedom from error mainly depends, I shall proceed in the next chapter to some applications of trigonometry, which will give a first view of the manner in which it is used in astronomy and other branches of physics.

# CHAPTER VIII.

## APPLICATIONS OF THE PRECEDING CHAPTERS.

(152.) The following question occurs several times in astronomy:— If $\tan\phi = m\tan\theta$, required the developement of $\phi$ in a series of terms which shall be functions of $m$ and $\theta$: the supposition being that $m$ is nearly unity, so that one value of $\phi$ is nearly $\theta$, or $\phi = \theta$ nearly is the solution to which better approximation is required. Let $\varepsilon^{\sqrt{-1}}$ be called $\eta$; then we have

$$\frac{1}{2\sqrt{-1}}\,\frac{\eta^{\phi}-\eta^{-\phi}}{\eta^{\phi}+\eta^{-\phi}} = \frac{m}{2\sqrt{-1}}\,\frac{\eta^{\theta}-\eta^{-\theta}}{\eta^{\theta}+\eta^{-\theta}} \quad \text{or} \quad \frac{\eta^{2\phi}-1}{\eta^{2\phi}+1} = m\,\frac{\eta^{2\theta}-1}{\eta^{2\theta}+1}$$

Now, $$\frac{1-\frac{x-1}{x+1}}{1+\frac{x-1}{x+1}} = \frac{1}{x} \qquad \frac{1-m\frac{x-1}{x+1}}{1+m\frac{x-1}{x+1}} = \frac{(1-m)x+1+m}{(1+m)x+1-m}$$

whence $$\eta^{-2\phi} = \frac{(1-m)\eta^{2\theta}+(1+m)}{(1+m)\eta^{2\theta}+(1-m)} = \frac{c\eta^{2\theta}+1}{\eta^{2\theta}+c} \text{ where } c = \frac{1-m}{1+m}$$

or $$\begin{aligned}\log\eta^{-2\phi} &= \log(1+c\eta^{2\theta}) - \log(1+c\eta^{-2\theta}) - \log\eta^{2\theta} \\ &= -\log\eta^{2\theta} + c(\eta^{2\theta}-\eta^{-2\theta}) - \frac{c^2}{2}\left(\eta^{4\theta}-\eta^{-4\theta}\right) \\ &\qquad + \frac{c^3}{3}\left(\eta^{6\theta}-\eta^{-6\theta}\right) - \text{\&c.}\end{aligned}$$

But $\eta^{-2\phi} = \varepsilon^{-2\phi\sqrt{-1}}$ $\quad \log\eta^{-2\phi} = -2\phi\sqrt{-1}+2\pi n\sqrt{-1}$, &c.

whence $$\begin{aligned}-2\phi\sqrt{-1}+2\pi n\sqrt{-1} &= -2\theta\sqrt{-1}-2\pi n'\sqrt{-1} \\ &\qquad + c\left(\varepsilon^{2\theta\sqrt{-1}} - \varepsilon^{-2\theta\sqrt{-1}}\right) - \text{\&c.}\end{aligned}$$

Divide both sides by $-2\sqrt{-1}$, and since $n+n'$ is simply any whole number positive or negative, which call $r$,

$$\phi - r\pi = \theta - 2c\sin 2\theta + \frac{2c^2}{2}\sin 4\theta - \frac{2c^3}{3}\sin 6\theta + \text{\&c.}$$

Since $\phi$ is to be the value nearest to $\theta$, we must have $r = 0$, or

$$\phi = \theta - 2c\sin 2\theta + \frac{2c^2}{2}\sin 4\theta - \frac{2c^3}{3}\sin 6\theta + \ \&c.$$

(153.) Let it now be proposed to expand the first side of the following equation in a series of the form of the second.

$$\frac{1}{1 + e\cos\theta} = a_0 + a_1\cos\theta + a_2\cos 2\theta + a_3\cos 3\theta + \dots\dots$$

Assume $$x^2 + y^2 = 1 \quad 2xy = e$$

$$1 + e\cos\theta = x^2 + 2xy \times \tfrac{1}{2}(\varepsilon^{\theta\sqrt{-1}} + \varepsilon^{-\theta\sqrt{-1}}) + y^2$$

$$= (x + y\varepsilon^{\theta\sqrt{-1}})(x + y\varepsilon^{-\theta\sqrt{-1}})$$

$$= x^2\left(1 + \frac{y}{x}\varepsilon^{\theta\sqrt{-1}}\right)\left(1 + \frac{y}{x}\varepsilon^{-\theta\sqrt{-1}}\right)$$

Let $$\frac{y}{x} = z \quad \varepsilon^{\theta\sqrt{-1}} = \omega.$$ Then,

$$\frac{x^2}{1 + e\cos\theta} = \frac{1}{1 + z\omega}\cdot\frac{1}{1 + z\omega^{-1}}$$

$$= (1 - z\omega + z^2\omega^2 - \dots)(1 - z\omega^{-1} + z^2\omega^{-2} - \dots)$$

$$= (1 + z^2 + z^4 + \dots) - (z + z^3 + \dots)(\omega + \omega^{-1})$$
$$+ (z^2 + z^4 + \dots)(\omega^2 + \omega^{-2}) - \dots\dots$$

$$= \frac{1}{1 - z^2}\left\{1 - 2z\cos\theta + 2z^2\cos 2\theta - 2z^3\cos 3\theta + \dots\right\}$$

Now, $$2x = \sqrt{1+e} + \sqrt{1-e} \quad 2y = \sqrt{1+e} - \sqrt{1-e}$$

$$z = \frac{y}{x} = \frac{\sqrt{1+e} - \sqrt{1-e}}{\sqrt{1+e} + \sqrt{1-e}} = \frac{1 - \sqrt{1-e^2}}{e} \quad (\textit{Algebra}, \text{p. } 119.)$$

$$x^2(1 - z^2) = x^2 - y^2 = \sqrt{1 - e^2}$$

$$\frac{1}{1 + e\cos\theta} = \frac{1}{\sqrt{1-e^2}}\cdot\left\{1 - 2\left(\frac{1 - \sqrt{1-e^2}}{e}\right)\cos\theta + 2\left(\frac{1 - \sqrt{1-e^2}}{e}\right)^2\cos 2\theta - \dots\right\}$$

If we take the other sign for $\sqrt{1-e^2}$ throughout, we have a developement which is an algebraical equivalent of the first side, but which is divergent when the one just found is convergent. It appears also that this form of developement is not arithmetical, unless $e$ be less than 1. We shall now develope the series in powers of $e$, so as to include every term up to the third power of $e$. Firstly we have (rejecting every term above the third power)

$(1-e^2)^{-\frac{1}{2}} = 1+\frac{1}{2}e^2+\ldots\ldots \quad \sqrt{1-e^2} = 1-\frac{1}{2}e^2-\frac{1}{8}e^4- \ldots$

$\frac{1-\sqrt{1-e^2}}{e} = \frac{1}{2}e + \frac{1}{8}e^3 + \ldots\ldots$ whence the series is

$$(1+\tfrac{1}{2}e^2)\Big(1-2(\tfrac{1}{2}e+\tfrac{1}{8}e^3)\cos\theta+2(\tfrac{1}{2}e+\tfrac{1}{8}e^3)^2\cos 2\theta$$
$$-2(\tfrac{1}{2}e+\tfrac{1}{8}e^3)^3\cos 3\theta + \ldots\ldots\Big)$$

Develope the second factor, rejecting all powers of $e$ above the third, which gives

$$(1+\tfrac{1}{2}e^2)\left(1-\frac{4e+e^3}{4}\cos\theta+\frac{e^2}{2}\cos 2\theta-\frac{e^3}{4}\cos 3\theta\right)$$

Multiply, rejecting $e^4$, &c. which gives

$$1+\tfrac{1}{2}e^2-\frac{4e+3e^3}{4}\cos\theta+\frac{e^2}{2}\cos 2\theta-\frac{e^3}{4}\cos 3\theta$$

Multiply both sides by $1-e^2$ with similar rejection, which gives

$$\frac{1-e^2}{1+e\cos\theta} = 1-\frac{e^2}{2}-\frac{4e-e^3}{4}\cos\theta+\frac{e^2}{2}\cos 2\theta-\frac{e^3}{4}\cos 3\theta$$

(154.) We might also obtain the preceding result as follows (*Algebra*, p. 161).

$$\frac{1-e^2}{1+e\cos\theta} = (1-e^2)\,(1-e\cos\theta+e^2\cos^2\theta-e^3\cos^3\theta), \text{ rejecting } e^4, \text{ \&c.}$$
$$= 1-e^2-(e-e^3)\cos\theta+e^2\cos^2\theta-e^3\cos^3\theta$$
$$= 1-e^2-(e-e^3)\cos\theta+e^2(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta)-e^3(\tfrac{3}{4}\cos\theta+\tfrac{1}{4}\cos 3\theta)$$

which, reduced, gives the same result as before.

(155.) THEOREM. While $\theta$ varies from 0 to $2\pi$, or from $\alpha$ to $\alpha+2\pi$, every expression of the form $a\sin\theta+b\sin 2\theta+\ldots.$ or $a\tan\theta+b\tan 2\theta+\ldots.$ &c. has as many positive as negative values; that is, for every positive value which the series has for one value of $\theta$, there is a negative value which it has for another, numerically equal to the positive value.

This theorem depends upon another, namely, that in the same revolution there is always a second angle $\theta'$ to any given angle $\theta$, such that

$$\sin\theta' = -\sin\theta,\ \sin 2\theta' = -\sin 2\theta,\ \sin 3\theta' = -\sin 3\theta \ldots. \textit{ ad inf.}$$
$$\tan\theta' = -\tan\theta,\ \tan 2\theta' = -\tan 2\theta,\ \tan 3\theta' = -\tan 3\theta \ldots. \textit{ ad inf.}$$

The angle in question is $2\pi-\theta$, as appears from (44). It is, therefore, evident that $\mathrm{F}\,\theta$ being either series mentioned, $\mathrm{F}\,(2\pi-\theta)$

$= -\mathrm{F}\theta$. The cosine has not a similar property, but is contained with the others in the following general theorem, of which the one just given is a more simple equivalent as regards the sine and tangent.

THEOREM. If $\theta$ be an aliquot part of $2\pi$ and $\mathrm{F}\theta = \sin\theta$ or $\cos\theta$, &c.

$$\mathrm{F}\theta + \mathrm{F}(2\theta) + \mathrm{F}(3\theta) + \cdots\cdots\cdots + \mathrm{F}(2\pi \text{ or } n\theta) = 0$$

This is already proved for the sine and tangent, since the series may be made from the beginning and end, into one of terms such as $\mathrm{F}x + \mathrm{F}(2\pi - x)$, which are severally $= 0$; and the middle term, if there be one, is $\mathrm{F}(\pi) = 0$, and the last term $\mathrm{F}(2\pi) = 0$. For the cosine, if there be a middle term, $\mathrm{F}(\pi)$, that is, if $n$ be even, all the series preceding $\mathrm{F}\pi$ may be arranged in terms of the form $\mathrm{F}x + \mathrm{F}(\pi - x)$, each of which $= 0$; and all from $\mathrm{F}(\pi + \theta)$ to $\mathrm{F}(2\pi - \theta)$, the last but one, may be arranged in terms of the form $\mathrm{F}(\pi + x) + \mathrm{F}(2\pi - x)$, which are severally $= 0$. And the remaining terms, $\mathrm{F}\pi$ and $\mathrm{F}2\pi$, give $\mathrm{F}\pi + \mathrm{F}2\pi = 0$. When there is not a middle term $\mathrm{F}\pi$, or when $n$ is odd, we must make use of a general demonstration derived from the method in (143), by which it it is proved that

$$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots\ldots + \cos n\theta = \frac{\cos\theta + \cos n\theta - 1 - \cos(n+1)\theta}{2(1 - \cos\theta)}$$

the second side of which becomes 0 when $n\theta = 2\pi$.

Similarly, in (143), we have the series

$$\sin\theta + \sin 2\theta + \sin 3\theta + \ldots\ldots + \sin n\theta = \frac{\sin\theta + \sin n\theta - \sin(n+1)\theta}{2(1 - \cos\theta)}$$

which becomes $= 0$ when $n\theta = 2\pi$. *And the same, in both cases, if* $n\theta = 2m\pi$.

Hence it appears, that if we have such an expression as $a\sin\theta + b\sin 2\theta + \ldots\ldots$, and if we substitute for $\theta$ successively $\frac{2\pi}{n}\ \frac{4\pi}{n}\ \ldots\ldots$ up to $2\pi$, the sum of each of the sets of terms corresponding to $a\sin\theta$, to $b\sin 2\theta$, &c., is $= 0$, whence the following result. If we take angles uniformly distributed in value between 0 and $2\pi$, the sum of the negative values of $a\sin\theta + b\sin 2\theta + \ldots\ldots$ will be equal to the sum of the positive values; and the same of $a\cos\theta + b\cos 2\theta + \ldots\ldots$; more briefly expressed thus: the *mean* *value* of either of the preceding expressions is *nothing*. If, then, we have an expression such as $a_0 + a_1 \sin\theta + a_2 \sin 2\theta + \ldots.$ the sum of $n$ values, the angles of which are uniformly distributed through four right angles, will be $n a_0 + 0$; that is, $a_0$, or $a_0$ is the *mean value* of the series. Hence an infinite number of ways of determining a magnitude which oscillates on one side and the other of a given magnitude: such as is almost every magnitude which is considered in astronomy.

(156.) We shall now inquire into the method of forming the product of two series such as $a_0 + a_1 \cos\theta + a_2 \cos 2\theta + \ldots..$ and $b_0 + b_1 \cos\theta + b_2 \cos 2\theta + \ldots...$ Let $\Sigma\phi(n)$ be the abbreviation of the series whose terms are the values of $\phi n$ for all whole numbers from 0 to $\propto$. Then the preceding series are denoted by $\Sigma a_n \cos n\theta$ and $\Sigma b_n \cos n\theta$; their sum is $\Sigma(a_n + b_n)\cos n\theta$, and its mean value $a_0 + b_0$; their difference $\Sigma(a_n - b_n)\cos n\theta$, and its mean value $a_0 - b_0$. Their product is $\Sigma a_n b_m \cos n\theta \cos m\theta$, for every possible simultaneous pair of values of $m$ and $n$, both being positive whole numbers. It is evident, moreover, that $\Sigma(p_n + q_n) = \Sigma p_n + \Sigma q_n$. If, then, we give the general term

$$a_n b_m \cos n\theta \cos m\theta \text{ its value } \tfrac{1}{2} a_n b_m \cos(m+n)\theta + \tfrac{1}{2} a_n b_m \cos(m-n)\theta$$

we have

$$\Sigma a_n \cos n\theta \times \Sigma b_m \cos m\theta = \tfrac{1}{2}\Sigma a_n b_m \cos(m+n)\theta + \tfrac{1}{2}\Sigma a_n b_m \cos(m-n)\theta$$

To find the co-efficient of a given cosine, $\cos p\theta$, in the product, we must inquire in how many ways $m+n$ may be made to give $p$, and $m-n$ either $p$ or $-p$, for $\cos(-p\theta) = \cos p\theta$. Let us, then, first ask for the *mean value* of the product, or the term independent of $\theta$. The only way in which $m+n$ gives nothing ($m$ and $n$ always positive whole numbers, 0 included) is when $m = 0$ $n = 0$, which contributes $\frac{1}{2} a_0 b_0$ to the mean value in question. But $m - n = 0$ whenever $m = n$, which contributes $\frac{1}{2} a_0 b_0 + \frac{1}{2} a_1 b_1 + \frac{1}{2} a_2 b_2 + \ldots.$ *ad inf.*; so that the mean value of the product is $a_0 b_0 + \frac{1}{2} a_1 b_1 + \frac{1}{2} a_2 b_2 + \ldots..$ The student might perhaps think that the mean value ought to have been $a_0 b_0$, and may reason thus: If there be a magnitude which, one time with another, is $a_0$, and a second which is $b_0$, the product is as often above as below $a_0 b_0$, and the latter should therefore be the mean value. But this is not true, as the most simple instance will shew. The average of 7 and 3 is 5; that of 4 and 12 is 8; the *product of the averages* is 40; the *average of the products* is 34½. In the series deduced in (153), the mean value of the square of $1 \div (1 + e\cos\theta)$ is, by the preceding rule, $a_0^2 + \frac{1}{2}a_1^2 + \frac{1}{2}a_2^2 + \dots$

or $$\frac{1}{1-e^2} + \frac{1}{2}\frac{4z^2}{1-e^2} + \frac{1}{2}\frac{4z^4}{1-e^2} + \dots$$

or $$\frac{1}{1-e^2} + \frac{2z^2}{1-e^2} \cdot \frac{1}{1-z^2} \quad \text{or} \quad (1-e^2)^{-\frac{3}{2}}$$

(157.) I now ask for the term of the product which contains $\cos 5\theta$. The number of ways in which $m+n$ can be made $= 5$ is finite, namely,

$$n=0 \ m=5; \ n=1 \ m=4; \ n=2 \ m=3;$$
$$n=3 \ m=2; \ n=4 \ m=1; \ n=5 \ m=0$$

which contributes

$$\tfrac{1}{2}(a_0b_5 + a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1 + a_5b_0)$$

to the co-efficient. Now all the ways of making $m-n=5$ or $-5$, are infinite in number, giving

$$m=5, \ n=0, \ \text{or} \ m=0, \ n=5; \ m=6, \ n=1,$$
$$\text{or} \ m=1, \ n=6; \ m=7, \ n=2, \ \text{or} \ m=2, \ n=7, \ \text{\&c.}$$

which contribute

$$\tfrac{1}{2}(a_0b_5 + a_5b_0) + \tfrac{1}{2}(a_1b_6 + a_6b_1) + \tfrac{1}{2}(a_2b_7 + a_7b_2) + \dots\dots$$

The co-efficient is therefore

$$\text{the half of} \left\{ \begin{array}{l} \phantom{+}a_0b_5 + a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1 + a_5b_0 \\ +a_0b_5 + a_1b_6 + a_2b_7 + a_3b_8 + a_4b_9 + \dots\dots \\ +a_5b_0 + a_6b_1 + a_7b_2 + a_8b_3 + a_9b_4 + \dots\dots \end{array} \right\}$$

Similarly, the co-efficient of $\cos n\theta$ is

$$\text{the half of} \left\{ \begin{array}{l} \phantom{+}a_0b_n + a_1b_{n-1} + \dots + a_nb_0 \\ +a_0b_n + a_1b_{n+1} + \dots + a_nb_0 + a_{n+1}b_1 + \dots \end{array} \right.$$

The student should now actually multiply some terms of $\Sigma a_n \cos n\theta$ and $\Sigma b_m \cos m\theta$, and thus produce the result here condensed, in a more expanded form.

(158.) The product $\Sigma a_n \sin n\theta \times \Sigma b_m \sin m\theta$ must be developed in a series of cosines: shew how the several terms may be deduced, and compare them with those of the last. The product $\Sigma a_n \sin n\theta \times \Sigma b_m \cos m\theta$ must be expanded in a series both of sines and cosines: proceed in the same way.

# CHAPTER IX.

## MISCELLANEOUS ADDITIONS TO THE PRECEDING CHAPTERS.

In this chapter I propose to touch slightly on several points which it is desirable the student should consider, but not necessary, so far as Trigonometry is concerned, that he should enter to any great depth.

(159.) **Problem.** Required a solution of any equation of the second degree which has possible roots, by help of the trigonometrical tables.

Let the equation be reduced to the form $x^2+2ax+b=0$, or $x^2+2ax-b=0$, $a$ being either positive or negative, and $b$ being positive. Under one or other of these forms every equation can be reduced.

1. $x^2+2ax+b=0$ gives $x=-a\pm\sqrt{a^2-b}$

Assume $b=a^2\sin^2\theta$, or find $\theta$ from $\sin\theta=\sqrt{b}\div a$; which can be done, for, the roots being possible, $a$ is greater than $\sqrt{b}$.

$$x=-a\pm\sqrt{a^2-a^2\sin^2\theta}=-a(1\mp\cos\theta)$$

and $-2a\sin^2\dfrac{\theta}{2}$ and $-2a\cos^2\dfrac{\theta}{2}$ are the roots required.

2. $x^2+2ax-b=0$ gives $x=-a\pm\sqrt{a^2+b}$

Assume $b=a^2\tan^2\theta$, or find $\theta$ from $\tan\theta=\sqrt{b}\div a$, which can be done, as the tangent of an angle may have any value.

$$x=-a(1\mp\sec\theta)=-a\frac{\cos\theta\pm1}{\cos\theta}=-\sqrt{b}\,\frac{\cos\theta\pm1}{\sin\theta}$$

and $\sqrt{b}.\tan\dfrac{\theta}{2}$ and $-\sqrt{b}\cot\dfrac{\theta}{2}$ are the roots.

N.B. If $a$ be negative, as in $x^2-2x-2=0$, it will be more convenient to solve $x^2+2x-2=0$, and to change the sign of the roots of the latter. (*Algebra*, Chapter V.)

Examples (with obvious roots for verification.)

$x^2+3x+2=0$

$a=1\cdot5 \qquad b=2$

| | |
|---|---|
| log $b$ | 2)·3010300 |
| | ·1505150 |
| log $a$ | ·1760913 |
| | $\bar{1}$·9744237 |
| | 10 |
| L sin 70° 31′ 43″·5 | 9·9744237 |

| | |
|---|---|
| log sin 35° 15′ 51″·8 | 9·7614394—10 |
| | 2 |
| | 19·5228788—20 |
| log $a$ | ·1760913 |
| | ·3010300 |
| log 1·000000 | 20·0000001—20 |

| | |
|---|---|
| log cos 35° 15′ 51″·8 | 9·9119544—10 |
| | 2 |
| | 19·8239088—20 |
| | ·1760913 |
| | ·3010300 |
| log 2·000000 | 20·3010301—20 |

Roots, $-1$ and $-2$.

$x^2+x-12=0$

$a=\cdot5 \qquad b=12$

| | |
|---|---|
| log $b$ | 2)1·0791812 |
| | ·5395906 |
| log $a$ | $\bar{1}$·6989700 |
| | 0·8406206 |
| | 10 |
| L.tan 81° 47′ 12″·5 | 10·8406206 |

| | |
|---|---|
| log tan 40° 53′ 36″·3 | 9·9375310—10 |
| log $\sqrt{b}$ | ·5395906 |
| log 3·000000 | 10·4771216—10 |

| | |
|---|---|
| log cot 40° 53′ 36″·3 | 0·0624690 |
| log $\sqrt{b}$ | ·5395906 |
| log 4·000000 | 0·6020596 |

Roots, 3 and $-4$.

(160.) On the preceding there is a remark to be made, in continuation of (120). Having found one root of the first equation to be $-2a\sin^2\frac{1}{2}\theta$, we perceive that at the point where $\theta$ was first introduced, in the expression $\sin\theta$, we might, whatever $\theta$ may be, have substituted $\pi-\theta$. This shews us that in the result, since $-2a\sin^2\frac{1}{2}\theta$ is a root, $-2a\sin^2\frac{1}{2}(\pi-\theta)$, or $-2a\cos^2\frac{1}{2}\theta$ is another root; that is, *the* other root, since there can be but two. We shall illustrate this subject by giving a method of solving an equation of the third degree. Let the equation be

$$x^3+3hx^2+3lx+m=0$$

Assume $x = y - k$, which gives

$$y^3 - (3k^2 - 3l)\,y + 2k^3 - 3kl + m = 0$$

Let $k^2 - l = p$, $2k^3 - 3kl + m = q$, whence $q = 3py - y^3$

But (59.) writing $1 - s^2$ for $c^2$, we have $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$; if, then, we assume $3py = 3\,\mathrm{A}\sin\theta$, and $y^3 = 4\,\mathrm{A}\sin^3\theta$, we have $q = \mathrm{A}\sin 3\theta$. But the first two equations give $\mathrm{A} = 2\sqrt{p^3}$, whence the equation can be solved if

$$\sin 3\theta = \sqrt{\frac{q^2}{4p^3}} \quad \text{for then} \quad y = 2\sqrt{p}\,.\sin\theta$$

This requires that $q^2$ should not exceed $4p^3$, in which case the first equation is rational. But $3\theta \pm 2\pi$ has the same sine as $3\theta$, whence, by the same process, we discover that the three following expressions are values of $y$, which make $q = 3py - y^3$.

$$2\sqrt{p}.\sin\left(\theta - \frac{2\pi}{3}\right) \quad 2\sqrt{p}\sin\theta, \quad 2\sqrt{p}\sin\left(\theta + \frac{2\pi}{3}\right) \ldots. \text{(A)}$$

(161.) If for $\varepsilon^{3\theta\sqrt{-1}}$ we write $z$, we find

$$\sin 3\theta = \frac{1}{2\sqrt{-1}}\left\{z - \frac{1}{z}\right\} = \sqrt{\frac{q^2}{4p^3}}$$

$$z = \frac{\sqrt{-q^2} \pm \sqrt{4p^3 - q^2}}{2\sqrt{p^3}}$$

If for $z$ we take either of these values, $\frac{1}{z}$ is the other with its sign changed. Let us assume, then,

$$2\sqrt{p^3}z = \sqrt{4p^3 - q^2} + \sqrt{-q^2} \quad \frac{2\sqrt{p^3}}{z} = \sqrt{4p^3 - q^2} - \sqrt{-q^2}$$

Extract the cube root of both sides, and we have

$$2^{\frac{1}{3}}p^{\frac{1}{2}}z^{\frac{1}{3}} \text{ or } 2^{\frac{1}{3}}p^{\frac{1}{2}}\varepsilon^{\theta\sqrt{-1}} = \sqrt[3]{\sqrt{4p^3 - q^2} + \sqrt{-q^2}}$$

$$2^{\frac{1}{3}}p^{\frac{1}{2}}z^{-\frac{1}{3}} \text{ or } 2^{\frac{1}{3}}p^{\frac{1}{2}}\varepsilon^{-\theta\sqrt{-1}} = \sqrt[3]{\sqrt{4p^3 - q^2} - \sqrt{-q^2}}$$

$$y = 2p^{\frac{1}{2}}\sin\theta = \frac{p^{\frac{1}{2}}}{\sqrt{-1}}\left(\varepsilon^{\theta\sqrt{-1}} - \varepsilon^{-\theta\sqrt{-1}}\right)$$

$$= \frac{1}{2^{\frac{1}{3}}\sqrt{-1}}\left(\sqrt[3]{\sqrt{4p^3 - q^2} + \sqrt{-q^2}} - \sqrt[3]{\sqrt{4p^3 - q^2} - \sqrt{-q^2}}\right)$$

which, since $\sqrt{-1} = \sqrt[3]{-\sqrt{-1}}$, and $-\sqrt[3]{a} = \sqrt[3]{-a}$, is

$$= \sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}-p^3}} + \sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}-p^3}} \quad \ldots\ldots \text{(B)}$$

This is the formula known by the name of Cardan, for the solution of $y^3 - 3py + q = 0$. It may be verified as follows:—Assume $y = u + v$, which gives

$$u^3 + v^3 + 3uv(u+v) - 3p(u+v) + q = 0$$

Let $uv = p$, then $u^3 + v^3 + q = 0$, or $\begin{Bmatrix} u^6 + qu^3 + p^3 = 0 \\ v^6 + qv^3 + p^3 = 0 \end{Bmatrix}$

whence $u^3$ and $v^3$ are the two roots of the last equation, treated as of the second degree. The expression (B) for $y$ is then easily deduced.

The utility of this result (which is of great historical importance) is not considerable, and I shall give the following only as exercises for the student.

1. The expressions (A) and (B) cannot both have possible forms.

2. If 1, $k$, and $k^2$ (106) be the cube roots of unity, and if A and B be the arithmetical cube roots of $u^3$ and $v^3$; then $u + v$, considered alone, may have the following forms

$$\mathrm{A}+\mathrm{B}, \quad \mathrm{A}+\mathrm{B}k, \quad \mathrm{A}+\mathrm{B}k^2, \quad \mathrm{B}+\mathrm{A}k, \quad \mathrm{B}+\mathrm{A}k^2,$$
$$\mathrm{B}k+\mathrm{A}k^2, \quad \mathrm{A}k+\mathrm{B}k^2,$$

none of which (from $uv = p$) are admissible, except

$$\mathrm{A}+\mathrm{B}, \quad \mathrm{A}k+\mathrm{B}k^2, \quad \mathrm{A}k^2+\mathrm{B}k,$$

which are the three roots of the equation, or values of $y$.

3. The expressions (A) are possible when *all* the roots are possible; and (B) when one only is possible.

(162.) The equation $x^{2n} - 2\cos\theta \,.\, x^n + 1$, is solved by every value of $x$ which solves $x^2 - 2\cos\left(\frac{\theta + 2m\pi}{n}\right) . x + 1 = 0$: for (72)

$$2\cos\phi = x + \frac{1}{x} \quad \text{gives} \quad 2\cos n\phi = x^n + \frac{1}{x^n}.$$

Shew from this that the roots of the given equation are found by giving successive whole values to $m$ in the formula

$$\cos\left(\frac{\theta + 2m\pi}{n}\right) + \sin\left(\frac{\theta + 2m\pi}{n}\right) . \sqrt{-1}$$

and shew, as in (103), that there cannot be more than $n$ roots.

(163.) Any quantity of the form $a+b\sqrt{-1}$ can be reduced to the form $r(\cos\theta+\sin\theta\sqrt{-1})$. The conditions evidently are, that $\tan\theta=b\div a$, and $r=\sqrt{a^2+b^2}$. Hence it is easy to shew, that all functions of $a+b\sqrt{-1}$, &c. may be reduced to the same form: thus (115) we have

$$(a+b\sqrt{-1})(a'+b'\sqrt{-1})=rr'\left\{\cos(\theta+\theta')+\sin(\theta+\theta')\sqrt{-1}\right\}$$
$$(a+b\sqrt{-1})^n = r^n\left\{\cos n\theta+\sin n\theta\sqrt{-1}\right\}$$

We may express $a+b\sqrt{-1}$ in the form $r\varepsilon^{\theta\sqrt{-1}}$, which is the most compendious method of expressing any algebraical quantity whatsoever, and allows of $r$ being supposed positive. A negative quantity is expressed by making $\theta$ any odd multiple of $\pi$ (102); a positive quantity by making $\theta$ any even multiple of $\pi$: a quantity of the form $+b\sqrt{-1}$ or $-b\sqrt{-1}$, by making $\theta$ of the form $(4m+1)\frac{\pi}{2}$ or $(4m+3)\frac{\pi}{2}$, and one of the form $a+b\sqrt{-1}$ by a value of $\theta$, which is no whole multiple whatsoever of a right angle.

(164.) Among possible expressions, those of the form $A\cos(a\theta+\alpha)$ or $A\sin(a\theta+\alpha)$, are such that the sum or difference of any of them is always reducible to the same form, whatever the values of A and $\alpha$ may be, provided only that $a$ always remains the same. The two forms are not essentially different, for

$$A\sin(a\theta+\alpha) \quad\text{is}\quad A\cos\left(a\theta+\overline{a-\frac{\pi}{2}}\right)$$

To prove the theorem, observe that

$$A\cos(a\theta+\alpha)+A'\cos(a\theta+\alpha') = (A\cos\alpha+A'\cos\alpha')\cos a\theta$$
$$-(A\sin\alpha+A'\sin\alpha')\sin a\theta$$

Assume

$$A\cos\alpha+A'\cos\alpha' = L\cos\lambda \quad A\sin\alpha+A'\sin\alpha' = L\sin\lambda$$

or
$$L=\sqrt{A^2+A'^2+2AA'\cos(\alpha-\alpha')},$$
$$\tan\lambda=\frac{A\sin\alpha+A'\sin\alpha'}{A\cos\alpha+A'\cos\alpha'}$$

whence $A\cos(a\theta+\alpha)+A'\cos(a\theta+\alpha') = L\cos(a\theta+\lambda.)$

(165.) I shall conclude this chapter with some examples of a process known by the name of successive substitution. Let us suppose any function of $x$, $\phi x$, and that, beginning with a value of $x$, say $a$, we form $\phi a = b$, $\phi b = c$, $\phi c = e$, &c. Or, agreeably to the notation of (122), suppose that we form $\phi a$, $\phi^2 a$, $\phi^3 a$, &c. Then the limit towards which we approach, if we approach any limit at all, must be a solution of $\phi x = x$.

For instance, take the function $1 + \frac{1}{2}x$: begin with $x = 0$, and successive substitution then gives

$$1,\ \ 1+\tfrac{1}{2},\ \ 1+\tfrac{1}{2}(1\tfrac{1}{2}) \text{ or } 1\tfrac{3}{4},\ \ 1+\tfrac{1}{2}(1\tfrac{3}{4}) \text{ or } 1\tfrac{7}{8},\ \ \text{\&c. \&c.}$$

the limit is 2, which is the solution of $1 + \frac{1}{2}x = x$.

To prove this generally, suppose it found that we can bring the results of successive substitutions as near as we please by carrying the process far enough. Let $k$ and $k + z$ be the results of two successive substitutions: then, by hypothesis, $k + z = \phi k$. The smaller $z$ is, the more nearly does $k$ solve the equation $\phi x = x$. Let $l$ be the limit, and let $k = l + \delta$, where $\delta$ may be as small as we please. Then $\phi l + \delta + z = \phi(l + \delta)$, which being true for values of $\delta$ and $z$ as small as we please, gives (*Algebra*, page 157) $\phi l = l$.

(166.) As an example, let the function be $20 + \sin x$, where 20 means 20 degrees, and $\sin x$ is to be reckoned as a fraction of a degree. We begin with $x = 0$, which gives $20^\circ$; then $20^\circ + \sin 20^\circ$ is $20^\circ{\cdot}342$, or $20^\circ\, 20'\tfrac{1}{2}$, and $20^\circ + \sin(20^\circ\, 20'\tfrac{1}{2})$ is $20^\circ{\cdot}347$; which is a near approximation to the solution of $x = 20 + \sin x$ where 1 means one degree.

(167.) Successive substitutions, finite in number, will sometimes give theorems by which the logarithmic tables may be examined in many parts at once. For instance,

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} \quad \text{or} \quad 2 \sin \frac{x}{2} = \sin x \,.\, \sec \frac{x}{2}$$

$$4 \sin \frac{x}{4} = 2 \sin \frac{x}{2} \sec \frac{x}{4} = \sin x \sec \frac{x}{2} \sec \frac{x}{4}$$

$$8 \sin \frac{x}{8} = 2 \sin \frac{x}{2} \sec \frac{x}{4} \sec \frac{x}{8} = \sin x \sec \frac{x}{2} \sec \frac{x}{4} \sec \frac{x}{8}$$

and so on; whence it follows that

$$2^n \sin \frac{x}{2^n} = \sin x \,.\, \sec \frac{x}{2} \sec \frac{x}{4} \ldots\ldots \sec \frac{x}{2^n}$$

$$\text{or} \qquad \cos \frac{x}{2} \,.\, \cos \frac{x}{4} \,.\, \cos \frac{x}{8} \ldots\ldots \cos \frac{x}{2^n} = \sin x \div 2^n \sin \frac{x}{2^n}$$

If $n$ increase without limit, the limit of the last dividor is $x$; whence

$$\cos\frac{x}{2}\ \cos\frac{x}{4}\ \cos\frac{x}{8}\ \cos\frac{x}{16}\ \ldots\ \textit{ad inf.}\ \text{has the limit}\ \frac{\sin x}{x}$$

This affords an easy method of verifying the value of $\pi$, as follows: assume $x = \frac{\pi}{2}$, we have then

$$\cos\frac{\pi}{4} \cdot \cos\frac{\pi}{8} \cdot \cos\frac{\pi}{16} \ldots\ldots\ldots = \frac{2}{\pi}$$

| | | | | |
|---|---|---|---|---|
| log cos | 45° | 0′ | 0″ | 9·8494850 — 10 |
| .... | 22 | 30 | 0 | 9·9656153 — 10 |
| .... | 11 | 15 | 0 | 9·9915739 — 10 |
| .... | 5 | 37 | 30 | 9 9979037 — 10 |
| .... | 2 | 48 | 45 | 9·9994766 — 10 |
| .... | 1 | 24 | 22·5 | 9·9998692 — 10 |
| .... | 0 | 42 | 11·3 | 9 9999673 — 10 |
| .... | 0 | 21 | 5·7 | 9·9999918 — 10 |
| .... | 0 | 10 | 32·9 | 9·9999980 — 10 |
| .... | 0 | 5 | 16·5 | 9·9999995 — 10 |
| .... | 0 | 2 | 38·3 | 9·9999999 — 10 |
| | | | | 109·8038802 — 110 |
| | | | | ·3010300 |
| log 3·141593 | | | | ·4971498 |

I should, however, recommend the student not to proceed further in the subject of developement of trigonometrical forms, until he is able to apply the Differential Calculus, without which the theory of series will always be very incomplete.

THE END.

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